Chandra-Furst-Lipton

Yaël Dillies

September 30, 2024

Chapter 1

Multidimensional corners

Let G be a finite abelian group whose size we will denote N. Let $d \geq 2$ a natural number.

Definition 1.1 (Forgetting a coordinate). *For an index $i : [d]$, we define*

$$\text{forget}_i : G^d \to G^{\{j:[d] \mid j \neq i\}} \tag{1.1}$$
$$x \mapsto j \mapsto x_j \tag{1.2}$$

Definition 1.2 (Forbidden pattern). *We say a tuple $(a_1, \dots, a_d) : (G^d)^d$ is a* **forbidden pattern with tip** $v : G^d$ *if*

$$a_{i,j} = v_j$$

for all i, j distinct. We also simply say $(a_1, \dots, a_d)$ is a **forbidden pattern** *if it is a forbidden pattern with tip v for some v.*

Definition 1.3 (Multidimensional corner).

A **multidimensional corner** *in d dimensions is a tuple of the form $(x, x + ce_1, \dots, x + ce_d)$ for some $x : G^d$ and $c : G$, where ce_i is the vector of all zeroes except in position i where it is c. Such a corner is said to be* **trivial** *if $c = 0$.*

Definition 1.4 (Corner-free number).

*The d-***dimensional corner-free** *number of G, denoted $r_d(G)$ is the size of the largest set A in G^d such that A doesn't contain a non-trivial corner.*

Definition 1.5 (Corner-coloring number).

*The d-***dimensional corner-coloring number** *of G, denoted $\chi_d(G)$, is the smallest number of colors one needs to color G^d such that no non-trivial d-dimensional corner is monochromatic.*

Lemma 1.6 (Lower bound on the corner-coloring number).

$$r_d(G)\chi_d(G) \geq N^d$$

Proof. Find a coloring of G^d in $\chi_d(G)$ colors without non-trivial monochromatic d-dimensional corners. The coloring partitions G^d into $\chi_d(G)$ sets of size at most $r_d(G)$. □

Lemma 1.7 (Upper bound on the corner-coloring number).

$$r_d(G)\chi_d(G) \leq 2dN^d \log N$$

Proof. Find A a corner-free set of density $\alpha = r_d(G)/N^d$. If we pick $m > d \log N/\alpha$ translates of A randomly, then the expected number of elements not covered by any translate is

$$N^d(1-\alpha)^m \leq \exp(dN - m\alpha) < 1$$

Namely, there is some collection of m translates of A that covers all of G^d. Since being corner-free is translation-invariant, this cover by translates gives a coloring in m colors without non-trivial monochromatic corners. So

$$\chi_d(G) \leq m \leq 2d \log N/\alpha = 2dN^d \log N/r_d(G)$$

if we set eg $m = \lfloor d \log N/\alpha \rfloor + 1$. □

Chapter 2

The NOF model

Let G be a finite abelian group whose size we will denote d. Let $d \geq 3$ a natural number.

Definition 2.1 (NOF protocol)**.** *A* **NOF protocol** P *consists of maps*

$$\text{strat} : [d] \to G^{d-1} \to \text{List Bool} \to \text{Bool} \tag{2.1}$$

$$\text{guess} : [d] \to G^{d-1} \to \text{List Bool} \to \text{Bool} \tag{2.2}$$

We will not make P part of any notation as it is usually fixed from the context.

Definition 2.2 (NOF broadcast)**.**
Given a NOF protocol P, the **NOF broadcast on input** $x : G^d$ *is inductively defined by*

$$\text{broad}(x) : \mathbb{N} \to \text{List Bool} \tag{2.3}$$

$$0 \mapsto [] \tag{2.4}$$

$$t + 1 \mapsto \text{strat}_{t\%d}(\text{forget}_{t\%d}(x), \text{broad}(x, t)) :: \text{broad}(x, t) \tag{2.5}$$

Lemma 2.3 (Length of a broadcast)**.** *For every NOF protocol P, every input $x : G^d$ and every time t,* $\text{broad}(x, t)$ *has length t.*

Proof. Induction on t. □

Definition 2.4 (Valid NOF protocol)**.**
Given a function $F : G^d \to$ Bool, the NOF protocol P is **valid in F at time t on input x** *if all participants correctly guess $F(x)$, namely if*

$$\text{guess}_i(\text{forget}_i(x), \text{broad}(x, t)) = F(x)$$

for all $i : [d]$.

Definition 2.5 (The trivial protocol)**.**
For all F, we define the **trivial protocol** *by making participant i do "Send the t/d-th bit of the number of participant $i + 1$" and "Compute x_i from the binary representation given by participant $i - 1$, then compute $F(x)$".*

Lemma 2.6 (The trivial protocol is valid)**.**
For all F, the trivial protocol for F is valid in time $d \lceil \log_2 n \rceil$.

Proof. Obvious. □

Definition 2.7 (Deterministic complexity of a protocol).

The **communication complexity of a NOF protocol** P **for** F *is the smallest time* t *such that* P *is valid in* F *at time* t *on all inputs* x*, or* ∞ *if no such* t *exists.*

Definition 2.8 (Deterministic complexity of a function).

The **deterministic communication complexity of a function** F*, denoted* $D(F)$*, is the minimum of the communication complexity of* P *when* P *ranges over all NOF protocols.*

Lemma 2.9 (Trivial bound on the function complexity).

The communication complexity of any function F *is at most* $d \lceil \log_2 n \rceil$.

Proof.

The trivial protocol is a protocol valid in F in time $d \lceil \log_2 n \rceil$. □

Lemma 2.10 (The tip of a monochromatic forbidden pattern).

Given P *a NOF protocol and a time* t*, if* $(a_1, \dots, a_d)$ *is a forbidden pattern with tip* v *such that* $\mathrm{broad}(a_i, t)$ *equals some fixed broadcast history* b *for all* i*, then* $\mathrm{broad}(v, t) = b$ *as well.*

Proof.

Induction on t. TODO: Expand □

Chapter 3

Lower bound on the communication complexity of eval

Definition 3.1 (eval function)**.** *The* eval **function** *is defined by*

$$\mathrm{eval} : G^d \to \mathrm{Bool} \tag{3.1}$$

$$x \mapsto \begin{cases} 1 & \text{if } \sum_i x_i = 0 \\ 0 & \text{else} \end{cases} \tag{3.2}$$

Lemma 3.2 (Forbidden patterns project to multidimensional corners)**.**
If $(a_1, \ldots, a_d)$ *is a forbidden pattern such that* $\mathrm{eval}(a_i) = 1$ *for all* i*, then*

$$(\mathrm{forget}_i(a_1), \ldots, \mathrm{forget}_i(a_d))$$

is a multidimensional corner for all index i*.*

Proof. Let v be the tip of $(a_1, \ldots, a_d)$. Then, using that $\sum_k a_{j,k} = 0$ and $v_k = a_{j,k}$ for all $k \neq j$, we see that $v_j = a_{j,j} + \sum_k v_k$. This means that $(\mathrm{forget}_i(a_1), \ldots, \mathrm{forget}_i(a_d)$ is a multidimensional corner by setting $x = \mathrm{forget}_i(a_i)$ and $c = \sum_k v_k$. □

Lemma 3.3 (Monochromatic forbidden patterns are trivial)**.**
Given P *a NOF protocol valid in time* t *for* eval*, all monochromatic forbidden patterns are trivial.*

Proof.
Assume $(a_1, \ldots, a_d)$ is a monochromatic forbidden pattern with tip v, say $\mathrm{broad}(a_i, t) = b$ for all i. By Lemma 2.10, we also have $\mathrm{broad}(v, t) = b$. Since P is a valid NOF protocol for eval, we get $\mathrm{eval}(a_i) = \mathrm{eval}(v)$ for all i, meaning that $(a_1, \ldots, a_d) = (v, \ldots, v)$ is trivial. □

Theorem 3.4 (Lower bound for $D(\mathrm{eval})$ in terms of $\chi_d(G)$)**.**

$$D(\mathrm{eval}) \geq \lceil \log_2 \chi_d(G) \rceil$$

Proof.
Let P be a protocol valid in time t for eval. By Lemma 3.3, $\mathrm{broad}(\cdot, t)$ is a coloring of $\{x \mid \sum_i x_i = 0\}$ in at most 2^t colors (since t bits were broadcasted) such that all monochromatic forbidden patterns are trivial. By Lemma 3.2, this yields a coloring of G^{d-1} in at most 2^t colors such all monochromatic corners are trivial. Hence $2^t \geq \chi_d(G)$, as wanted. □

Corollary 3.5 (Lower bound for $D(\text{eval})$ in terms of $r_d(G)$)**.**

$$D(\text{eval}) \geq d \log_2 \frac{N}{r_d(G)}$$

Proof.

Putting Theorem 3.4 and Lemma 1.6 together, we get

$$D(\text{eval}) \geq \left\lceil \log_2 \frac{2dN^d \log N}{r_d(G)} \right\rceil \geq d \log_2 \frac{N}{r_d(G)}$$

□

Chapter 4

Upper bound on the deterministic communication complexity of eval

Definition 4.1 (The non-monochromatic protocol)**.**
Given a coloring $c : \{x \mid \operatorname{eval} x = 1\} \to [C]$*, writing* a_i *the vector whose* j*-th coordinate is* x_j *except when* $j = i$ *in which case it is* $-\sum_{j \neq i} x_j$*, we define the* **non-monochromatic protocol for** c *by making participant* i *do "Send the* t/d*-th bit of* $c(a_i)$ *until time* $\lceil \log_2 \chi_d(G) \rceil$*, then send* 1 *iff* $c(a_i)$ *agrees with the broadcast from time* 1 *to time* $\lceil \log_2 \chi_d(G) \rceil$ *read as a color" and "Send* 1 *iff the broadcasts from time* $\lceil \log_2 \chi_d(G) \rceil$ *to time* $\lceil \log_2 \chi_d(G) \rceil + d$ *were all* 1*".*

Lemma 4.2 (The non-monochromatic protocol is valid)**.**
If c *is a coloring such that all monochromatic forbidden patterns are trivial, then the non-monochromatic protocol for* c *is valid in time* $\lceil \log_2 \chi_d(G) \rceil + d$*.*

Proof. We have

$$\text{answer is } 1 \iff \text{all } a_i \text{ have the same color} \iff \text{all } a_i \text{ are equal} \iff \sum_i x_i = 0$$

where the first equivalence holds by definition, the second one holds by assumption and the third one holds since the a_i form a forbidden pattern. □

Theorem 4.3 (Upper bound for $D(\text{eval})$ in terms of $\chi_d(G)$)**.**

$$D(\text{eval}) \leq \lceil \log_2 \chi_d(G) \rceil + d$$

Proof.
Using Lemma 3.2, find some coloring c of $\{x \mid \sum_i x_i = 0\}$ in $\chi_d(G)$ colors such that all monochromatic forbidden patterns are trivial. Then Lemma 4.2 tells us that the non-monochromatic protocol for c is valid in time $\lceil \log_2 \chi_d(G) \rceil + d$. □

Corollary 4.4 (Upper bound for $D(\text{eval})$ in terms of $r_d(G)$)**.**

$$D(\text{eval}) \leq 2d \log_2 \frac{N}{r_d(G)}$$

Proof.

Putting Theorem 4.3 and Lemma 1.7 together, we get

$$D(\text{eval}) \leq \left\lceil \log_2 \frac{2dN^d \log N}{r_d(G)} \right\rceil \leq 2d \log_2 \frac{N}{r_d(G)}$$

□

Chapter 5

Randomised complexity of eval

Definition 5.1 (Randomised complexity of a protocol).

The **communication complexity of a randomised NOF protocol** P **for** F **with error** ϵ *is the smallest time* t *such that*

$$\mathbb{P}(x \mid P \textit{ is not valid at time } t) \leq \epsilon$$

or ∞ *if no such* t *exists.*

Definition 5.2 (Randomised complexity of a function).

The **randomised communication complexity of a function** F **with error** ϵ*, denoted* $R_\epsilon(F)$*, is the minimum of the randomised communication complexity of* P *when* P *ranges over all randomised NOF protocols.*

Definition 5.3 (The randomised equality testing protocol for eval).

The **randomised equality testing protocol for** eval *has domain* $\Omega := (\mathrm{Bool}^d)^{\lceil \log_2 \epsilon^{-1} \rceil}$ *with the uniform measure and is defined by making participant* i *do "Compute*

$$a_{i,k} = \sum_{j \neq i} \omega_{j,k} x_j$$

and send the sum of the digits of $a_{i,t/d}$ *mod* 2 *at time* t*" and "Guess* 1 *iff the sum of the digits of* $\omega_i x_i$ + *what participant* i *said is* 0 *modulo* 2*".*

Lemma 5.4 (The randomised equality testing protocol for eval is valid).

The randomised equality testing protocol is valid for eval *at time* $2d$*.*

Proof. If $\mathrm{eval}(x) = 1$, then the protocol guesses correctly. Else it errors with probability

$$2^{-\lceil \log_2 \epsilon^{-1} \rceil} \leq \epsilon$$

□

Theorem 5.5 (The randomised complexity of eval is constant).

$$R_\epsilon(\mathrm{eval}) \leq 2d \lceil \log_2 \epsilon^{-1} \rceil$$

Proof.

By Lemma 5.4, the randomised equality testing protocol is valid for eval at time $2d$. □