

Arithmetic Progressions - Almost Periodicity

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Chapter 1

Almost-Periodicity

Lemma 1.1 (Marcinkiewicz-Zygmund inequality). *Let $m \geq 1$. If $f : G \rightarrow \mathbb{R}$ is such that $\mathbb{E}_x f(x) = 0$ and $|f(x)| \leq 2$ for all x then*

$$\mathbb{E}_{x_1, \dots, x_n} \left| \sum_{i=1}^n f(x_i) \right|^{2m} \leq (4mn)^m.$$

Proof. Let S be the left-hand side. Since $0 = \mathbb{E}_y f(y)$ we have, by the triangle inequality, and Hölder's inequality,

$$S = \mathbb{E}_{x_1, \dots, x_n} \left| \sum_i f(x_i) - \mathbb{E}_{y_i} f(y_i) \right|^{2m} \quad (1.1)$$

$$= \mathbb{E}_{x_1, \dots, x_n} \left| \mathbb{E}_{y_i} \left(\sum_i f(x_i) - f(y_i) \right) \right|^{2m} \quad (1.2)$$

$$\leq \mathbb{E}_{x_1, \dots, y_n} \left| \sum_i f(x_i) - f(y_i) \right|^{2m} \quad (1.3)$$

Changing the role of x_i and y_i makes no difference here, but multiplies the i summand by $\{-1, +1\}$, and therefore for any $\epsilon_i \in \{-1, +1\}$,

$$S \leq \mathbb{E}_{x_1, \dots, y_n} \left| \sum_i \epsilon_i (f(x_i) - f(y_i)) \right|^{2m}.$$

In particular, if we sample $\epsilon_i \in \{-1, +1\}$ uniformly at random, then

$$S \leq \mathbb{E}_{\epsilon_i} \mathbb{E}_{x_1, \dots, y_n} \left| \sum_i \epsilon_i (f(x_i) - f(y_i)) \right|^{2m}.$$

We now change the order of expectation and consider the expectation over just ϵ_i , viewing the $f(x_i) - f(y_i) = z_i$, say, as fixed quantities. For any z_i we can expand $\mathbb{E}_{\epsilon_i} |\sum_i \epsilon_i z_i|^{2m}$ and then bound it from above, using the triangle inequality and $|z_i| \leq 4$, by

$$4^{2m} \sum_{k_1 + \dots + k_n = 2m} \binom{2m}{k_1, \dots, k_n} \left| \mathbb{E} \epsilon_1^{k_1} \dots \epsilon_n^{k_n} \right|.$$

The inner expectation vanishes unless each k_i is even, when it is trivially one. Therefore the above quantity is exactly

$$\sum_{l_1+\dots+l_n=m} \binom{2m}{2l_1, \dots, 2l_n} \leq m^m n^m,$$

since for any $l_1 + \dots + l_n = m$,

$$\binom{2m}{2l_1, \dots, 2l_n} \leq m^m \binom{m}{l_1, \dots, l_n}.$$

This can be seen, for example, by writing both sides out using factorials, yielding

$$\frac{(2m)!}{(2l_1)!\dots(2l_n)!} \leq \frac{(2m)!}{2^m m!} \frac{m!}{l_1!\dots l_n!} \leq m^m \frac{m!}{l_1!\dots l_n!}.$$

□

Lemma 1.2 (Complex-valued Marcinkiewicz-Zygmund inequality). *Let $m \geq 1$. If $f : G \rightarrow \mathbb{C}$ is such that $\mathbb{E}_x f(x) = 0$ and $|f(x)| \leq 2$ for all x then*

$$\mathbb{E}_{x_1, \dots, x_n} \left| \sum_{i=1}^n f(x_i) \right|^{2m} \leq (8mn)^m.$$

Proof. Test. □

Lemma 1.3. *Let $\epsilon > 0$ and $m \geq 1$. Let $A \subseteq G$ and $f : G \rightarrow \mathbb{C}$. If $k \geq 64m\epsilon^{-2}$ then the set*

$$L = \{\vec{a} \in A^k : \left\| \frac{1}{k} \sum_{i=1}^k f(x - a_i) - \mu_A * f \right\|_{2m} \leq \epsilon \|f\|_{2m}\}.$$

has size at least $|A|^k/2$.

Proof. Note that if $a \in A$ is chosen uniformly at random then, for any fixed $x \in G$,

$$\mathbb{E} f(x - a) = \frac{1}{|A|} \sum_{a \in A} f(x - a) = \frac{1}{|A|} \mathbf{1}_A * f(x) = \mu_A * f(x).$$

Therefore, if we choose $a_1, \dots, a_k \in A$ independently uniformly at random, for any fixed $x \in G$ and $1 \leq i \leq k$, the random variable $f(x - a_i) - f * \mu_A(x)$ has mean zero. By the Marcinkiewicz-Zygmund inequality Lemma 1.1, therefore,

$$\begin{aligned} \mathbb{E} \left| \frac{1}{k} \sum_i f(x - a_i) - f * \mu_A(x) \right|^{2m} &\leq \\ (16m/k)^m k^{-1} \mathbb{E} \sum_i &|f(x - a_i) - f * \mu_A(x)|^{2m}. \end{aligned}$$

We now sum both sides over all $x \in G$. By the triangle inequality, for any fixed $1 \leq i \leq k$ and $a_i \in A$,

$$\begin{aligned} \sum_{x \in G} |f(x - a_i) - f * \mu_A(x)|^{2m} &\leq 2^{2m-1} \sum_{x \in G} |f(x - a_i)|^{2m} + \sum_{x \in G} |f * \mu_A(x)|^{2m} \\ &\leq 2^{2m-1} (\|f\|_{2m}^{2m} + \|f * \mu_A\|_{2m}^{2m}). \end{aligned}$$

We note that $\|\mu_A\|_1 = \frac{1}{|A|} \sum_{x \in A} 1_A(x) = |A| / |A| = 1$, and hence by Young's inequality, $\|f * \mu_A\|_{2m} \leq \|f\|_{2m}$, and so

$$\sum_{x \in G} |f(x - a_i) - f * \mu_A(x)|^{2m} \leq 2^{2m} \|f\|_{2m}^{2m}.$$

It follows that

$$\mathbb{E}_{a_1, \dots, a_k \in A} \left\| \frac{1}{k} \sum_i \tau_{a_i} f - f * \mu_A \right\|_{2m}^{2m} \leq (64m/k)^m \|f\|_{2m}^{2m}.$$

In particular, if $k \geq 64\epsilon^{-2}m$ then the right-hand side is at most $(\frac{\epsilon}{2} \|f\|_{2m})^{2m}$ as required. \square

Lemma 1.4. *Let $A \subseteq G$ and $f : G \rightarrow \mathbb{C}$. Let $\epsilon > 0$ and $m \geq 1$ and $k \geq 1$. Let*

$$L = \{\vec{a} \in A^k : \left\| \frac{1}{k} \sum_{i=1}^k f(x - a_i) - \mu_A * f \right\|_{2m} \leq \epsilon \|f\|_{2m}\}.$$

If $t \in G$ is such that $\vec{a} \in L$ and $\vec{a} + (t, \dots, t) \in L$ then

$$\|\tau_t(\mu_A * f) - \mu_A * f\|_{2m} \leq 2\epsilon \|f\|_{2m}.$$

Proof. Test. \square

Lemma 1.5. *Let $A \subseteq G$ and $k \geq 1$ and $L \subseteq A^k$. Then there exists some $\vec{a} \in L$ such that*

$$\#\{t \in G : \vec{a} + (t, \dots, t) \in L\} \geq \frac{|L|}{|A + S|^k} |S|.$$

Proof. Test. \square

Theorem 1.6 (L_p almost periodicity). *Let $\epsilon \in (0, 1]$ and $m \geq 1$. Let $K \geq 2$ and $A, S \subseteq G$ with $|A + S| \leq K|A|$. Let $f : G \rightarrow \mathbb{C}$. There exists $T \subseteq G$ such that*

$$|T| \geq K^{-512m\epsilon^{-2}} |S|$$

such that for any $t \in T$ we have

$$\|\tau_t(\mu_A * f) - \mu_A * f\|_{2m} \leq \epsilon \|f\|_{2m}.$$

Proof. Test. \square

Theorem 1.7 (L_∞ almost periodicity). *Let $\epsilon \in (0, 1]$. Let $K \geq 2$ and $A, S \subseteq G$ with $|A + S| \leq K|A|$. Let $B, C \subseteq G$. Let $\eta = \min(1, |C|/|B|)$. There exists $T \subseteq G$ such that*

$$|T| \geq K^{-4096 \lceil \mathcal{L}\eta \rceil \epsilon^{-2}} |S|$$

such that for any $t \in T$ we have

$$\|\tau_t(\mu_A * 1_B * \mu_C) - \mu_A * 1_B * \mu_C\|_\infty \leq \epsilon.$$

Proof. Let T be as given in 1.6 with $f = 1_B$ and $m = \lceil \mathcal{L}\eta \rceil$ and $\epsilon = \epsilon/e$. (The size bound on T follows since $e^2 \leq 8$.) Fix $t \in T$ and let $F = \tau_t(\mu_A * 1_B) - \mu_A * 1_B$. We have, for any $x \in G$,

$$(\tau_t(\mu_A * 1_B * \mu_C) - \mu_A * 1_B * \mu_C)(x) = F * \mu_C(x) = \sum_y F(y) \mu_C(x - y) = \sum_y F(y) \mu_{x-C}(y).$$

By Hölder's inequality, this is (in absolute value), for any $m \geq 1$,

$$\|F\|_{2m} \|\mu_{x-C}\|_{1+\frac{1}{2m-1}}.$$

By the construction of T the first factor is at most $\frac{\epsilon}{e} \|1_B\|_{2m} = \frac{\epsilon}{e} |B|^{1/2m}$. We have by calculation

$$\|\mu_{x-C}\|_{1+\frac{1}{2m-1}} = |x - C|^{-1/2m} = |C|^{-1/2m}.$$

Therefore we have shown that

$$\|\tau_t(\mu_A * 1_B * \mu_C) - \mu_A * 1_B * \mu_C\|_\infty \leq \frac{\epsilon}{e} (|C|/|B|)^{-1/2m}.$$

The claim now follows since, by choice of m ,

$$(|C|/|B|)^{-1/2m} \leq e$$

(dividing into cases as to whether $\eta = 1$ or not). \square

Theorem 1.8. *Let $\epsilon \in (0, 1]$ and $k \geq 1$. Let $K \geq 2$ and $A, S \subseteq G$ with $|A + S| \leq K|A|$. Let $B, C \subseteq G$. Let $\eta = \min(1, |C|/|B|)$. There exists $T \subseteq G$ such that*

$$|T| \geq K^{-4096\lceil \mathcal{L}\eta \rceil k^2 \epsilon^{-2}} |S|$$

such that

$$\|\mu_T^{(k)} * \mu_A * 1_B * \mu_C - \mu_A * 1_B * \mu_C\|_\infty \leq \epsilon.$$

Proof. Let T be as in Theorem 1.7 with ϵ replaced by ϵ/k . Note that, for any $x \in G$,

$$\mu_T^{(k)} * \mu_A * 1_B * \mu_C(x) = \frac{1}{|T|^k} \sum_{t_1, \dots, t_k \in T} \tau_{t_1 + \dots + t_k} \mu_A * 1_B * \mu_C(x).$$

It therefore suffices (by the triangle inequality) to show, for any fixed $x \in G$ and $t_1, \dots, t_k \in T$, that with $F = \mu_A * 1_B * \mu_C$, we have

$$|\tau_{t_1 + \dots + t_k} F(x) - F(x)| \leq \epsilon.$$

This follows by the triangle inequality applied k times if we knew that, for $1 \leq l \leq k$,

$$|\tau_{t_1 + \dots + t_l} F(x) - \tau_{t_1 + \dots + t_{l-1}} F(x)| \leq \epsilon/k.$$

We can write the left-hand side as

$$|\tau_{t_1 + \dots + t_l} F(x) - \tau_{t_1 + \dots + t_{l-1}} F(x)| = |\tau_{t_l} F(x - t_1 - \dots - t - l - 1) - F(x - t_1 - \dots - t - l - 1)|.$$

The right-hand side is at most

$$\|\tau_{t_l} F - F\|_\infty$$

and we are done by choice of T . \square

Chapter 2

Chang's lemma

Definition 2.1 (Dissociation). *We say that $A \subseteq G$ is dissociated if, for any $m \geq 1$, and any $x \in G$, there is at most one $A' \subset A$ of size $|A'| = m$ such that*

$$\sum_{a \in A'} a = x.$$

Lemma 2.2 (Rudin's exponential inequality). *If the discrete Fourier transform of $f : G \rightarrow \mathbb{C}$ has dissociated support, then*

$$\mathbb{E} \exp(\Re f) \leq \exp\left(\frac{\|f\|_2^2}{2}\right)$$

It follows that

$$\mathbb{E}_x e^{|f(x)|} \leq 2e^{\|f\|_2^2/2}.$$

Proof. Using the convexity of $t \mapsto e^{tx}$ (for all $x \geq 0$ and $t \in [-1, 1]$) we have

$$e^{tx} \leq \cosh(x) + t \sinh(x).$$

It follows (taking $x = |z|$ and $t = \Re(z)/|z|$) that, for any $z \in \mathbb{C}$,

$$e^{\Re z} \leq \cosh|z| + \Re(z/|z|) \sinh|z|.$$

In particular, if $c_\gamma \in \mathbb{C}$ with $|c_\gamma| = 1$ is such that $\widehat{f}(\gamma) = c_\gamma |\widehat{f}(\gamma)|$, then

$$\begin{aligned} e^{\Re f(x)} &= \exp\left(\Re \sum_{\gamma \in \Gamma} \widehat{f}(\gamma) \gamma(x)\right) \\ &= \prod_{\gamma \in \Gamma} \exp\left(\Re \widehat{f}(\gamma) \gamma(x)\right) \\ &\leq \prod_{\gamma \in \Gamma} \left(\cosh|\widehat{f}(\gamma)| + \Re c_\gamma \gamma(x) \sinh|\widehat{f}(\gamma)|\right). \end{aligned}$$

Therefore

$$\mathbb{E}_x e^{\Re f(x)} \leq \mathbb{E}_x \prod_{\gamma \in \Gamma} \left(\cosh|\widehat{f}(\gamma)| + \Re c_\gamma \gamma(x) \sinh|\widehat{f}(\gamma)|\right).$$

Using $\Re z = (z + \bar{z})/2$ the product here can be expanded as the sum of

$$\prod_{\gamma \in \Gamma_2} \frac{c_\gamma}{2} \prod_{\gamma \in \Gamma_3} \frac{\bar{c}_\gamma}{2} \left(\prod_{\gamma \in \Gamma_1} \cosh |\widehat{f}(\gamma)| \right) \left(\prod_{\gamma \in \Gamma_2 \cup \Gamma_3} \sinh |\widehat{f}(\gamma)| \right) \left(\sum_{\gamma \in \Gamma_2} \gamma - \sum_{\lambda \in \Gamma_3} \lambda \right) (x)$$

as $\Gamma_1 \sqcup \Gamma_2 \sqcup \Gamma_3 = \Gamma$ ranges over all partitions of Γ into three disjoint parts. Using the definition of dissociativity we see that

$$\sum_{\gamma \in \Gamma_2} \gamma - \sum_{\lambda \in \Gamma_3} \lambda \neq 0$$

unless $\Gamma_2 = \Gamma_3 = \emptyset$. In particular summing this term over all $x \in G$ gives 0. Therefore the only term that survives averaging over x is when $\Gamma_1 = \Gamma$, and so

$$\mathbb{E}_x e^{\Re f(x)} \leq \prod_{\gamma \in \Gamma} \cosh |\widehat{f}(\gamma)|.$$

The conclusion now follows using $\cosh(x) \leq e^{x^2/2}$ and $\sum_{\gamma \in \Gamma} |\widehat{f}(\gamma)|^2 = \|f\|_2^2$. The second conclusion follows by applying it to $f(x)$ and $-f(x)$ and using

$$e^{|y|} \leq e^y + e^{-y}.$$

□

Lemma 2.3 (Rudin's inequality). *If the discrete Fourier transform of $f : G \rightarrow \mathbb{C}$ has dissociated support and $p \geq 2$ is an integer, then $\|f\|_p \leq 4\sqrt{pe}\|f\|_2$.*

Proof. It is enough to show that $\|\Re f\|_p \leq 2\sqrt{pe}\|f\|_2$ as then

$$\|f\|_p \leq \|\Re f\|_p + \|i\Im f\|_p = \|\Re f\|_p + \|\Re(-if)\|_p \leq 4\sqrt{pe}\|f\|_2$$

If $f = 0$, the result is obvious. So assume $f \neq 0$. $\|f\|_2 > 0$, so WLOG $\|f\|_2 = \sqrt{p}$.

Rudin's exponential inequality for f becomes $\mathbb{E} \exp |\Re f| \leq 2 \exp(\frac{p}{2}) = (2\sqrt{e})^p$. Using $\frac{x^p}{p!} \leq e^x$ for positive x , we get

$$\frac{\|\Re f\|_p^p}{p^p} \leq \frac{\|\Re f\|_p^p}{p!} = \mathbb{E} \frac{|\Re f|^p}{p!} \leq \mathbb{E} \exp |\Re f|$$

Rearranging, $\|\Re f\|_p \leq 2p\sqrt{e} = 2\sqrt{pe}\|f\|_2$. □

Definition 2.4 (Large spectrum). *Let G be a finite abelian group and $f : G \rightarrow \mathbb{C}$. Let $\eta \in \mathbb{R}$. The η -large spectrum is defined to be*

$$\Delta_\eta(f) = \{\gamma \in \widehat{G} : |\widehat{f}(\gamma)| \geq \eta\|f\|_1\}.$$

Definition 2.5 (Weighted energy). *Let $\Delta \subseteq \widehat{G}$ and $m \geq 1$. Let $\nu : G \rightarrow \mathbb{C}$. Then*

$$E_{2m}(\Delta; \nu) = \sum_{\gamma_1, \dots, \gamma_{2m} \in \Delta} |\widehat{\nu}(\gamma_1 + \dots + \gamma_{2m})|.$$

Definition 2.6 (Energy). *Let G be a finite abelian group and $A \subseteq G$. Let $m \geq 1$. We define*

$$E_{2m}(A) = \sum_{a_1, \dots, a_{2m} \in A} 1_{a_1 + \dots + a_{2m} = 0}.$$

Lemma 2.7. *Let G be a finite abelian group and $f : G \rightarrow \mathbb{C}$. Let $\nu : G \rightarrow \mathbb{R}_{\geq 0}$ be such that whenever $|f| \neq 0$ we have $\nu \geq 1$. Let $\Delta \subseteq \Delta_\eta(f)$. Then, for any $m \geq 1$.*

$$\eta^{2m} \frac{\|f\|_1^2}{\|f\|_2^2} |\Delta|^{2m} \leq E_{2m}(\Delta; \nu).$$

Proof. By definition of $\Delta_\eta(f)$ we know that

$$\eta \|f\|_1 |\Delta| \leq \sum_{\gamma \in \Delta} |\widehat{f}(\gamma)|.$$

There exists some $c_\gamma \in \mathbb{C}$ with $|c_\gamma| = 1$ for all γ such that

$$|\widehat{f}(\gamma)| = c_\gamma \widehat{f}(\gamma) = c_\gamma \sum_{x \in G} f(x) \overline{\gamma(x)}.$$

Interchanging the sums, therefore,

$$\eta \|f\|_1 |\Delta| \leq \sum_{x \in G} f(x) \sum_{\gamma \in \Delta} c_\gamma \overline{\gamma(x)}.$$

By Hölder's inequality the right-hand side is at most

$$\left(\sum_x |f(x)| \right)^{1-1/m} \left(\sum_x |f(x)| \left| \sum_{\gamma \in \Delta} c_\gamma \overline{\gamma(x)} \right|^m \right)^{1/m}.$$

Taking m th powers, therefore, we have

$$\eta^m |\Delta|^m \|f\|_1 \leq \sum_x |f(x)| \left| \sum_{\gamma \in \Delta} c_\gamma \overline{\gamma(x)} \right|^m.$$

By assumption we can bound $|f(x)| \leq |f(x)| \nu(x)^{1/2}$, and therefore by the Cauchy-Schwarz inequality the right-hand side is bounded above by

$$\|f\|_2 \left(\sum_x \nu(x) \left| \sum_{\gamma \in \Delta} c_\gamma \overline{\gamma(x)} \right|^{2m} \right)^{1/2}.$$

Squaring and simplifying, we deduce that

$$\eta^{2m} |\Delta|^{2m} \frac{\|f\|_1^2}{\|f\|_2^2} \leq \sum_x \nu(x) \left| \sum_{\gamma \in \Delta} c_\gamma \overline{\gamma(x)} \right|^{2m}.$$

Expanding out the power, the right-hand side is equal to

$$\sum_x \nu(x) \sum_{\gamma_1, \dots, \gamma_{2m}} c_{\gamma_1} \cdots \overline{c_{\gamma_{2m}}} (\overline{\gamma_1} \cdots \gamma_{2m})(x).$$

Changing the order of summation this is equal to

$$\sum_{\gamma_1, \dots, \gamma_{2m}} c_{\gamma_1} \cdots \overline{c_{\gamma_{2m}}} \widehat{\nu}(\gamma_1 \cdots \gamma_{2m}).$$

The result follows by the triangle inequality. \square

Lemma 2.8. *Let G be a finite abelian group and $f : G \rightarrow \mathbb{C}$. Let $\Delta \subseteq \Delta_\eta(f)$. Then, for any $m \geq 1$.*

$$N^{-1} \eta^{2m} \frac{\|f\|_1^2}{\|f\|_2^2} |\Delta|^{2m} \leq E_{2m}(\Delta).$$

Proof. Apply Lemma 2.7 with $\nu \equiv 1$, and use the fact that $\sum_x \lambda(x)$ is N if $\lambda \equiv 1$ and 0 otherwise. \square

Lemma 2.9. *If $A \subseteq G$ and $m \geq 1$ then*

$$E_{2m}(A) = \sum_x 1_A^{(m)}(x)^2.$$

Proof. Expand out definitions. \square

Lemma 2.10. *If $A \subseteq G$ is dissociated then $E_{2m}(A) \leq (32em|A|)^m$.*

Proof. By Lemma 2.9 and Lemma 2.3

$$\begin{aligned} E_{2m}(A) &= \mathbb{E}_\gamma |\hat{1}_A(\gamma)|^{2m} \\ &= \|\hat{1}_A\|_{2m}^{2m} \\ &\leq (4\sqrt{2em})^{2m} \|\hat{1}_A\|_2^{2m} \\ &= (32em)^m \|1_A\|_2^{2m} \\ &= (32em)^m |A|^m \end{aligned}$$

\square

Lemma 2.11.

If $A \subseteq G$ contains no dissociated set with $\geq K+1$ elements, then there is $A' \subseteq A$ of size $|A'| \leq K$ such that

$$A \subseteq \left\{ \sum_{a \in A'} c_a a : c_a \in \{-1, 0, 1\} \right\}.$$

Proof. Let $A' \subseteq A$ be a maximal dissociated subset (this exists and is non-empty, since trivially any singleton is dissociated). We have $|A'| \leq K$ by assumption.

Let S be the span on the right-hand side. It is obvious that $A' \subseteq S$. Suppose that $x \in A \setminus A'$. Then $A' \cup \{x\}$ is not dissociated by maximality. Therefore there exists some $y \in G$ and two distinct sets $B, C \subseteq A' \cup \{x\}$ such that

$$\sum_{b \in B} b = y = \sum_{c \in C} c.$$

If $x \notin B$ and $x \notin C$ then this contradicts the dissociativity of A' . If $x \in B$ and $x \in C$ then we have

$$\sum_{b \in B \setminus x} b = y - x = \sum_{c \in C \setminus x} c,$$

again contradicting the dissociativity of A' . Without loss of generality, therefore, $x \in B$ and $x \notin C$. Therefore

$$x = \sum_{c \in C} c - \sum_{b \in B \setminus x} b$$

which is in the span as required. \square

Theorem 2.12 (Chang's lemma). *Let G be a finite abelian group and $f : G \rightarrow \mathbb{C}$. Let $\eta > 0$ and $\alpha = N^{-1} \|f\|_1^2 / \|f\|_2^2$. There exists some $\Delta \subseteq \Delta_\eta(f)$ such that*

$$|\Delta| \leq \lceil e\mathcal{L}(\alpha)\eta^{-2} \rceil$$

and

$$\Delta_\eta(f) \subseteq \left\{ \sum_{\gamma \in \Delta} c_\gamma \gamma : c_\gamma \in \{-1, 0, 1\} \right\}.$$

Proof. By Lemma 2.11 it suffices to show that $\Delta_\eta(f)$ contains no dissociated set with at least

$$K = \lceil e\mathcal{L}(\alpha)\eta^{-2} \rceil + 1$$

many elements. Suppose not, and let $\Delta \subseteq \Delta_\eta(f)$ be a dissociated set of size K . Then by Lemma 2.10 we have, for any $m \geq 1$,

$$E_{2m}(\Delta) \leq m!K^m.$$

On the other hand, by Lemma 2.8,

$$\eta^{2m}\alpha K^{2m} \leq E_{2m}(\Delta).$$

Rearranging these bounds, we have

$$K^m \leq m! \alpha^{-1} \eta^{-2m} \leq m^m \alpha^{-1} \eta^{-2m}.$$

Therefore $K \leq \alpha^{-1/m} m \eta^{-2}$. This is a contradiction to the choice of K if we choose $m = \mathcal{L}(\alpha)$, since $\alpha^{-1/m} \leq e$. \square

Chapter 3

Unbalancing

Lemma 3.1. *For any function $f : G \rightarrow \mathbb{R}$ and integer $k \geq 0$*

$$\mathbb{E}_x f \circ f(x)^k \geq 0.$$

Proof. Test. □

Lemma 3.2. *Let $\epsilon \in (0, 1)$ and $\nu : G \rightarrow \mathbb{R}_{\geq 0}$ be some probability measure such that $\widehat{\nu} \geq 0$. Let $f : G \rightarrow \mathbb{R}$. If $\|f \circ f\|_{p(\nu)} \geq \epsilon$ for some $p \geq 1$ then $\|f \circ f + 1\|_{p'(\nu)} \geq 1 + \frac{1}{2}\epsilon$ for $p' = 120\epsilon^{-1} \log(3/\epsilon)$.*

Proof. Up to gaining a factor of 5 in p' , we can assume that $p \geq 5$ is an odd integer. Since the Fourier transforms of f and ν are non-negative,

$$\mathbb{E}_x f^p = \widehat{\nu} * \widehat{f}^{(p)}(0_{\widehat{G}}) \geq 0.$$

It follows that, since $2 \max(x, 0) = x + |x|$ for $x \in \mathbb{R}$,

$$2\langle \max(f, 0), f^{p-1} \rangle_{\nu} = \mathbb{E}_x f^p + \langle |f|, f^{p-1} \rangle_{\nu} \geq \|f\|_{p(\nu)}^p \geq \epsilon^p.$$

Therefore, if $P = \{x : f(x) \geq 0\}$, then $\langle 1_P, f^p \rangle_{\nu} \geq \frac{1}{2}\epsilon^p$. Furthermore, if $T = \{x \in P : f(x) \geq \frac{3}{4}\epsilon\}$ then $\langle 1_{P \setminus T}, f^p \rangle_{\nu} \leq \frac{1}{4}\epsilon^p$, and hence by the Cauchy-Schwarz inequality,

$$\nu(T)^{1/2} \|f\|_{2p(\nu)}^p \geq \langle 1_T, f^p \rangle_{\nu} \geq \frac{1}{4}\epsilon^p.$$

On the other hand, by the triangle inequality

$$\|f\|_{2p(\nu)} \leq 1 + \|f + 1\|_{2p(\nu)} \leq 3,$$

or else we are done, with $p' = 2p$. Hence $\nu(T) \geq (\epsilon/3)^{3p}$. It follows that, for any $p' \geq 1$,

$$\|f + 1\|_{p'(\nu)} \geq \langle 1_T, |f + 1|^{p'} \rangle_{\nu}^{1/p'} \geq (1 + \frac{3}{4}\epsilon)(\epsilon/3)^{3p/p'}.$$

The desired bound now follows if we choose $p' = 24\epsilon^{-1} \log(3/\epsilon)p$, using $1 - x \leq e^{-x}$. □

Chapter 4

Dependent random choice

Lemma 4.1. *Let $p \geq 2$ be an even integer. Let $B_1, B_2 \subseteq G$ and $\mu = \mu_{B_1} \circ \mu_{B_2}$. For any finite set $A \subseteq G$ and function $f : G \rightarrow \mathbb{R}_{\geq 0}$ there exist $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$ such that*

$$\langle \mu_{A_1} \circ \mu_{A_2}, f \rangle \|1_A \circ 1_A\|_{p(\mu)}^p \leq 2 \langle (1_A \circ 1_A)^p, f \rangle_\mu$$

and

$$\min\left(\frac{|A_1|}{|B_1|}, \frac{|A_2|}{|B_2|}\right) \geq \frac{1}{4} |A|^{-2p} \|1_A \circ 1_A\|_{p(\mu)}^{2p}.$$

Proof. First note that the statement is trivially true (with $A_1 = B_1$ and $A_2 = B_2$, say) if $\|1_A \circ 1_A\|_{p(\mu)}^p = 0$. We can therefore assume this is $\neq 0$.

For $s \in G^p$ let $A_1(s) = B_1 \cap (A + s_1) \cap \dots \cap (A + s_p)$, and similarly for $A_2(s)$. Note that

$$\begin{aligned} \langle (1_A \circ 1_A)^p, f \rangle_\mu &= \sum_x \mu_{B_1} \circ \mu_{B_2}(x) (1_A \circ 1_A(x))^p f(x) \\ &= \sum_{b_1, b_2} \mu_{B_1}(b_1) \mu_{B_2}(b_2) 1_A \circ 1_A(b_1 - b_2)^p f(b_1 - b_2) \\ &= \sum_{b_1, b_2} \mu_{B_1}(b_1) \mu_{B_2}(b_2) \left(\sum_{t \in G} 1_{A+t}(b_1) 1_{A+t}(b_2) \right)^p f(b_1 - b_2) \\ &= \sum_{b_1, b_2} \mu_{B_1}(b_1) \mu_{B_2}(b_2) \sum_{s \in G^p} 1_{A_1(s)}(b_1) 1_{A_2(s)}(b_2) f(b_1 - b_2) \\ &= |B_1|^{-1} |B_2|^{-1} \sum_{s \in G^p} \langle 1_{A_1(s)} \circ 1_{A_2(s)}, f \rangle. \end{aligned}$$

In particular, applying this with $f \equiv 1$ we see that

$$\|1_A \circ 1_A\|_{p(\mu)}^p = |B_1|^{-1} |B_2|^{-1} \sum_s |A_1(s)| |A_2(s)|$$

and

$$\frac{\langle (1_A \circ 1_A)^p, f \rangle_\mu}{\|1_A \circ 1_A\|_{p(\mu)}^p} = \frac{\sum_s \langle 1_{A_1(s)} \circ 1_{A_2(s)}, f \rangle}{\sum_s |A_1(s)| |A_2(s)|} = \eta,$$

say. Let $M > 0$ be some parameter, and let

$$g(s) = \begin{cases} 1 & \text{if } 0 < |A_1(s)| |A_2(s)| < M^2 \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\sum_s g(s) |A_1(s)| |A_2(s)| < \sum_s M |A_1(s)|^{1/2} |A_2(s)|^{1/2}.$$

To see why, note first that each summand on the left-hand side is \leq the corresponding summand on the right-hand side, arguing by cases on whether $g(s) = 1$ or not. It therefore suffices to show that there exists some s such that the summand on the left-hand side is $<$ the corresponding summand on the right-hand side.

If $g(s) = 0$ for all s then choose some s such that $|A_1(s)| |A_2(s)| \geq M^2$ (this must exist or else $|A_1(s)| |A_2(s)| = 0$ for all s , but then $\|1_A \circ 1_A\|_{p(\mu)}^p = 0$ by the above calculation). Otherwise, choose some s such that $g(s) = 1$, and note that for this s we have, by definition of s ,

$$|A_1(s)| |A_2(s)| < M |A_1(s)|^{1/2} |A_2(s)|^{1/2}.$$

We now choose

$$M = \frac{1}{2} |A|^{-p} (|B_1| |B_2|)^{1/2} \|1_A \circ 1_A\|_{p(\mu)}^p,$$

so that, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \sum_s g(s) |A_1(s)| |A_2(s)| &< M \sum_s |A_1(s)|^{1/2} |A_2(s)|^{1/2} \\ &\leq M \left(\sum_s \sum_{x \in G} 1_{A_1(s)}(x) \right)^{1/2} \left(\sum_s \sum_{x \in G} 1_{A_2(s)}(x) \right)^{1/2} \\ &= M |A|^p (|B_1| |B_2|)^{1/2} \\ &= \frac{1}{2} \sum_s |A_1(s)| |A_2(s)| \end{aligned}$$

and so

$$\sum_s (1 - g(s)) |A_1(s)| |A_2(s)| > \frac{1}{2} \sum_s |A_1(s)| |A_2(s)|$$

whence

$$\sum_s \langle 1_{A_1(s)} \circ 1_{A_2(s)}, f \rangle = \eta \sum_s |A_1(s)| |A_2(s)| < 2\eta \sum_s |A_1(s)| |A_2(s)| (1 - g(s)).$$

In particular there must exist some s such that

$$\langle 1_{A_1(s)} \circ 1_{A_2(s)}, f \rangle < 2\eta |A_1(s)| |A_2(s)| (1 - g(s)).$$

We claim this s meets the requirements. The first is satisfied since the right-hand side is $\leq 2\eta |A_1(s)| |A_2(s)|$. The second is satisfied since the left-hand side is trivially ≥ 0 and hence such an s must satisfy $g(s) = 0$, whence either $|A_1(s)| |A_2(s)| \geq M^2$, that is,

$$|A_1(s)| |A_2(s)| \geq \frac{1}{4} |A|^{-2p} |B_1| |B_2| \|1_A \circ 1_A\|_{p(\mu)}^{2p},$$

or $|A_1(s)| |A_2(s)| = 0$, which can't happen because then the right-hand side is $= 0$.

The final bound now follows since $xy \leq \min(x, y)$ when $x, y \leq 1$. \square

Lemma 4.2. Let $\epsilon, \delta > 0$ and $p \geq \max(2, \epsilon^{-1} \log(2/\delta))$ be an even integer. Let $B_1, B_2 \subseteq G$, and let $\mu = \mu_{B_1} \circ \mu_{B_2}$. For any finite set $A \subseteq G$, if

$$S = \{x \in G : 1_A \circ 1_A(x) > (1 - \epsilon) \|1_A \circ 1_A\|_{p(\mu)}\},$$

then there are $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$ such that

$$\langle \mu_{A_1} \circ \mu_{A_2}, 1_S \rangle \geq 1 - \delta$$

and

$$\min\left(\frac{|A_1|}{|B_1|}, \frac{|A_2|}{|B_2|}\right) \geq \frac{1}{4} |A|^{-2p} \|1_A \circ 1_A\|_{p(\mu)}^{2p}.$$

Proof. Apply Lemma 4.1 with $f = 1_{G \setminus S}$. This produces some $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$ such that

$$\langle \mu_{A_1} \circ \mu_{A_2}, 1_{G \setminus S} \rangle \leq 2 \frac{\langle (1_A \circ 1_A)^p, 1_{G \setminus S} \rangle_\mu}{\|1_A \circ 1_A\|_{p(\mu)}^p}$$

and

$$\min\left(\frac{|A_1|}{|B_1|}, \frac{|A_2|}{|B_2|}\right) \geq \frac{1}{4} |A|^{-2p} \|1_A \circ 1_A\|_{p(\mu)}^{2p}.$$

It then suffices to note that

$$\langle \mu_{A_1} \circ \mu_{A_2}, 1_S \rangle = 1 - \langle \mu_{A_1} \circ \mu_{A_2}, 1_{G \setminus S} \rangle$$

and by definition of S we have

$$\langle (1_A \circ 1_A)^p, 1_{G \setminus S} \rangle_\mu \leq (1 - \epsilon)^p \|1_A \circ 1_A\|_{p(\mu)}^p \sum_x \mu(x) = (1 - \epsilon)^p \|1_A \circ 1_A\|_{p(\mu)}^p.$$

Now use the fact that $p \geq \epsilon^{-1} \log(2/\delta)$ together with the inequality $1 - x \leq e^{-x}$ to deduce that the right-hand side is $\leq \frac{\delta}{2} \|1_A \circ 1_A\|_{p(\mu)}^p$. \square

Corollary 4.3. Let $\epsilon, \delta > 0$ and $p \geq \max(2, \epsilon^{-1} \log(2/\delta))$ be an even integer and $\mu \equiv 1/N$. If $A \subseteq G$ has density α and

$$S = \{x : \mu_A \circ \mu_A(x) \geq (1 - \epsilon) \|\mu_A \circ \mu_A\|_{p(\mu)}\}$$

then there are $A_1, A_2 \subseteq G$ such that

$$\langle \mu_{A_1} \circ \mu_{A_2}, 1_S \rangle \geq 1 - \delta$$

and both A_1 and A_2 have density

$$\geq \frac{1}{4} \alpha^{2p}.$$

Proof. We apply Lemma 4.2 with $B_1 = B_2 = G$. It remains to note that

$$\|1_A \circ 1_A\|_{p(\mu)} \geq \|1_A \circ 1_A\|_{1(\mu)} = |A|^2/N.$$

\square

Chapter 5

Finite field model

Theorem 5.1. *If $A_1, A_2, S \subseteq \mathbb{F}_q^n$ are such that A_1 and A_2 both have density at least α then there is a subspace V of codimension*

$$\text{codim}(V) \leq 2^{27} \mathcal{L}(\alpha)^2 \mathcal{L}(\epsilon\alpha)^2 \epsilon^{-2}$$

such that

$$|\langle \mu_V * \mu_{A_1} * \mu_{A_2}, 1_S \rangle - \langle \mu_{A_1} * \mu_{A_2}, 1_S \rangle| \leq \epsilon.$$

Proof. (In this proof we write $G = \mathbb{F}_q^n$.) Let $k = \lceil \mathcal{L}(\epsilon\alpha/4) \rceil$. Note that $|A_1 + G| = |G| \leq \alpha^{-1}|A_1|$. Furthermore, $|A_2|/|S| \geq \alpha$. Therefore by Theorem 1.8 there exists some $T \subseteq G$ with

$$|T| \geq \exp(-2^{16} \mathcal{L}(\alpha)^2 k^2 \epsilon^{-2}) |S|$$

such that

$$\|\mu_T^{(k)} * \mu_{A_1} * \mu_{A_2} \circ 1_S - \mu_{A_1} * \mu_{A_2} \circ 1_S\|_\infty \leq \epsilon/4.$$

Let $\Delta = \Delta_{1/2}(\mu_T)$ and

$$V = \{x \in G : \gamma(x) = 1 \text{ for all } \gamma \in \Delta\}.$$

Note that

$$\langle \mu_V * \mu_{A_1} * \mu_{A_2}, 1_S \rangle = \langle \mu_V, \mu_{A_1} * \mu_{A_2} \circ 1_S \rangle = \frac{1}{|V|} \sum_{v \in V} \mu_{A_1} * \mu_{A_2} \circ 1_S(v)$$

and

$$\langle \mu_{A_1} * \mu_{A_2}, 1_S \rangle = \mu_{A_1} * \mu_{A_2} \circ 1_S(0).$$

Therefore

$$|\langle \mu_V * \mu_{A_1} * \mu_{A_2}, 1_S \rangle - \langle \mu_{A_1} * \mu_{A_2}, 1_S \rangle| \leq \frac{1}{|V|} \sum_{v \in V} |\mu_{A_1} * \mu_{A_2} \circ 1_S(v) - \mu_{A_1} * \mu_{A_2} \circ 1_S(0)|.$$

In particular it suffices to show that, for any $v \in V$,

$$|\mu_{A_1} * \mu_{A_2} \circ 1_S(v) - \mu_{A_1} * \mu_{A_2} \circ 1_S(0)| \leq \epsilon.$$

By the triangle inequality and construction of T , it suffices to show that

$$\left| \mu_T^{(k)} * \mu_{A_1} * \mu_{A_2} \circ 1_S(v) - \mu_T^{(k)} * \mu_{A_1} * \mu_{A_2} \circ 1_S(0) \right| \leq \epsilon/2.$$

By the Fourier transform we have, for any $x \in G$,

$$\mu_T^{(k)} * \mu_{A_1} * \mu_{A_2} \circ 1_S(x) = \frac{1}{N} \sum_{\gamma \in \widehat{G}} \widehat{\mu_T}(\gamma)^k \widehat{\mu_{A_1}}(\gamma) \widehat{\mu_{A_2}}(\gamma) \widehat{1_S}(\gamma) \gamma(x).$$

Therefore the left-hand side of the desired inequality is, by the triangle inequality, at most

$$\frac{1}{N} \sum_{\gamma \in \widehat{G}} |\widehat{\mu_T}(\gamma)|^k \left| \widehat{\mu_{A_1}}(\gamma) \widehat{\mu_{A_2}}(\gamma) \widehat{1_S}(\gamma) \right| |\gamma(v) - 1|.$$

By choice of $v \in V$ the summand vanishes when $\gamma \in \Delta$. When $\gamma \notin \Delta$ the summand is bounded above by

$$2^{1-k} \left| \widehat{\mu_{A_1}}(\gamma) \widehat{\mu_{A_2}}(\gamma) \widehat{1_S}(\gamma) \right|.$$

Therefore the left-hand side of the desired inequality is at most

$$2^{1-k} \frac{1}{N} \sum_{\gamma} \left| \widehat{\mu_{A_1}}(\gamma) \widehat{\mu_{A_2}}(\gamma) \widehat{1_S}(\gamma) \right| \leq 2^{1-k} |S| \frac{1}{N} \sum_{\gamma} |\widehat{\mu_{A_1}}(\gamma) \widehat{\mu_{A_2}}(\gamma)|$$

using the trivial bound $|\widehat{1_S}| \leq |S|$. By the Cauchy-Schwarz inequality the sum on the right is at most

$$\left(\sum_{\gamma} |\widehat{\mu_{A_1}}|^2 \right)^{1/2} \left(\sum_{\gamma} |\widehat{\mu_{A_2}}|^2 \right)^{1/2}.$$

By Parseval's identity this is at most α^{-1} . Therefore the desired inequality follows from

$$2^{1-k} |S| \frac{1}{N} \alpha^{-1} \leq 2^{1-k} \alpha^{-1} \leq \epsilon/2.$$

It remains to check the codimension of V . For this, let $\Delta' \subseteq \Delta$ be as provided by Chang's lemma, Lemma 2.12, so that

$$\Delta \subseteq \left\{ \sum_{\gamma \in \Delta'} c_{\gamma} \gamma : c_{\gamma} \in \{-1, 0, 1\} \right\}.$$

Let

$$W = \{x \in G : \gamma(x) = 1 \text{ for all } \gamma \in \Delta'\}.$$

It follows that $W \leq V$, so it suffices to bound the codimension of W . This we can bound trivially using the bound from Chang's lemma and the fact that $\mathcal{L}(\delta) = \log(e^2/\delta) \leq 2 + \log(1/\delta) \leq 4 \log(1/\delta)$, provided $\log(1/\delta) \geq 1$, so

$$|\Delta'| \leq \lceil 4e\mathcal{L}(\delta) \rceil \leq 2^7 \log(1/\delta),$$

where

$$\delta = |T|/N \geq \exp(-2^{16}\mathcal{L}(\alpha)^2 k^2 \epsilon^{-2}),$$

so

$$\text{codim}(V) \leq |\Delta'| \leq 2^{23} \mathcal{L}(\alpha)^2 k^2 \epsilon^{-2} \leq 2^{25} \mathcal{L}(\alpha)^2 \mathcal{L}(\epsilon\alpha/4)^2 \epsilon^{-2},$$

and now use $\mathcal{L}(\epsilon\alpha/4) \leq 2\mathcal{L}(\epsilon\alpha)$, say. \square

Lemma 5.2. *For any function $f : G \rightarrow \mathbb{C}$ and integer $k \geq 1$*

$$\|f * f\|_{2k} \leq \|f \circ f\|_{2k}.$$

Proof. To finish, similar trick to unbalancing. \square

Lemma 5.3. *For any function f with $\sum f(x) = 1$*

$$f * f - 1/N = (f - 1/N) * (f - 1/N).$$

Proof. Expand everything out. \square

Lemma 5.4. *For any function f with $\sum f(x) = 1$*

$$f \circ f - 1/N = (f - 1/N) \circ (f - 1/N).$$

Proof. Expand everything out. \square

Lemma 5.5. *Let $\epsilon > 0$ and $\mu \equiv 1/N$. If $A, C \subseteq G$, where C has density at least γ , and*

$$|N\langle \mu_A * \mu_A, \mu_C \rangle - 1| > \epsilon$$

then, if $f = (\mu_A - 1/N)$, $\|f \circ f\|_{p(\mu)} \geq \epsilon/2N$ for $p = 2\lceil \mathcal{L}(\gamma) \rceil$.

Proof. By Hölder's inequality, for any $p \geq 1$

$$\epsilon < |N\langle \mu_A * \mu_A - 1/N, \mu_C \rangle| \leq \|\mu_A * \mu_A - 1/N\|_p \gamma^{-1/p} N^{1-1/p}.$$

In particular if we choose $p = 2\lceil \mathcal{L}(\gamma) \rceil$ then $\gamma^{-1/p} \leq e^{1/2} \leq 2$ and so we deduce that, by Lemma 5.3,

$$\|f * f\|_p \geq \frac{1}{2}\epsilon N^{1/p-1}.$$

It remains to use Lemmas 5.3 and 5.4 and apply Lemma 5.2, and note that we can pass from the L^p norm to the $L^p(\mu)$ norm losing a factor of $N^{1/p}$. \square

Proposition 5.6. *Let $\epsilon \in (0, 1)$. If $A, C \subseteq \mathbb{F}_q^n$, where C has density at least γ , and*

$$|N\langle \mu_A * \mu_A, \mu_C \rangle - 1| > \epsilon$$

then there is a subspace V of codimension

$$\leq 2^{171}\epsilon^{-24}\mathcal{L}(\alpha)^4\mathcal{L}(\gamma)^4.$$

*such that $\|1_A * \mu_V\|_\infty \geq (1 + \epsilon/32)\alpha$.*

Proof.

By Lemma 5.5, if $f = \mu_A - 1/N$,

$$\|f \circ f\|_{p(\mu)} \geq \epsilon/2N,$$

where $p = 2\lceil \mathcal{L}(\gamma) \rceil \leq 4\mathcal{L}(\gamma)$. By Lemma 3.2 there exists some p' such that

$$p' \leq 128\epsilon^{-1} \log(96/\epsilon)\mathcal{L}(\gamma)$$

such that

$$\|f \circ f + 1/N\|_{p'(\mu)} \geq (1 + \epsilon/4)/N.$$

By Lemma 5.4 $f \circ f + 1/N = \mu_A \circ \mu_A$.

Let $q = 2\lceil p' + 2^8\epsilon^{-2} \log(64/\epsilon) \rceil$. By Corollary 4.3, there are A_1, A_2 , both of density $\geq \alpha^{2q}$ such that

$$\langle \mu_{A_1} \circ \mu_{A_2}, 1_S \rangle \geq 1 - \epsilon/32$$

where

$$S = \{x : \mu_A \circ \mu_A(x) \geq (1 - \epsilon/16) \|\mu_A \circ \mu_A\|_{q(\mu)}\}.$$

Since

$$\|\mu_A \circ \mu_A\|_{q(\mu)} \geq \|\mu_A \circ \mu_A\|_{p'(\mu)} \geq (1 + \epsilon/4)/N$$

we know

$$S \subseteq S' = \{x : \mu_A \circ \mu_A(x) \geq (1 + \epsilon/8)/N\}.$$

By Theorem 5.1 (applied with ϵ replaced by $\epsilon/32$) there is a subspace V of codimension

$$\leq 2^{37} \mathcal{L}(\alpha^{2q})^2 \mathcal{L}(\epsilon \alpha^{2q}/32)^2 \epsilon^{-2}$$

such that

$$\langle \mu_V * \mu_{A_1} \circ \mu_{A_2}, 1_{S'} \rangle \geq 1 - \frac{1}{16}\epsilon.$$

Using $\mathcal{L}(xy) \leq x^{-1}\mathcal{L}(y)$ we have

$$\mathcal{L}(\epsilon \alpha^{2q}/32) \leq 32\epsilon^{-1}\mathcal{L}(\alpha^{2q}),$$

and we also use $\mathcal{L}(x^y) \leq y\mathcal{L}(x)$ to simplify the codimension bound to

$$\leq 2^{51}q^4\mathcal{L}(\alpha)^4\epsilon^{-4}.$$

We further note that (using $\log x \leq x$ say)

$$q \leq 2^{10}p'\epsilon^{-2} \log(64/\epsilon) \leq 2^{30}\epsilon^{-5}\mathcal{L}(\gamma).$$

Therefore the desired codimension bound follows. Finally, by definition of S' , it follows that

$$\begin{aligned} (1 + \epsilon/32)/N &\leq ((1 + \epsilon/8)/N)(1 - \epsilon/16) \\ &\leq \langle \mu_V * \mu_{A_1} \circ \mu_{A_2}, \mu_A \circ \mu_A \rangle \\ &\leq \|\mu_V * \mu_A\|_\infty \|\mu_A * \mu_{A_2} \circ \mu_{A_1}\|_1 \\ &= \|\mu_V * 1_A\|_\infty |A|^{-1}, \end{aligned}$$

and the proof is complete. \square

Lemma 5.7. *If $A \subseteq G$ has no non-trivial three-term arithmetic progressions and G has odd order then*

$$\langle \mu_A * \mu_A, \mu_{2 \cdot A} \rangle = 1/|A|^2.$$

Proof. Expand out using definitions. \square

Theorem 5.8. *Let q be an odd prime power. If $A \subseteq \mathbb{F}_q^n$ with $\alpha = |A|/q^n$ has no non-trivial three-term arithmetic progressions then*

$$n \ll \mathcal{L}(\alpha)^9.$$

Proof.

Let $t \geq 0$ be maximal such that there is a sequence of subspaces $\mathbb{F}_q^n = V_0 \geq \dots \geq V_t$ and associated $A_i \subseteq V_i$ with $A_0 = A$ such that

1. for $0 \leq i \leq t$ there exists x_i such that $A_i \subseteq A - x_i$,

2. with $\alpha_i = |A_i| / |V_i|$ we have

$$\alpha_{i+1} \geq \frac{65}{64} \alpha_i$$

for $0 \leq i < t$, and

3.

$$\text{codim}(V_{i+1}) \leq \text{codim}(V_i) + O(\mathcal{L}(\alpha)^8)$$

for $0 \leq i < t$. (here the second summand should be replaced with whatever explicit codimension bound we get from the above).

Note this is well-defined since $t = 0$ meets the requirements, and this process is finite and $t \ll \mathcal{L}(\alpha)$, since $\alpha_i \leq 1$ for all i . Therefore

$$\text{codim}(V_t) \ll \mathcal{L}(\alpha)^9.$$

Suppose first that

$$|V_t| \langle \mu_{A_t} * \mu_{A_t}, \mu_{2 \cdot A_t} \rangle < 1/2.$$

In this case we now apply Proposition 5.6 to $A_t \subseteq V_t$ with $\epsilon = 1/2$ (note that $N = |V_t|$ and all inner product, μ etc, are relative to the ambient group V_t now). Therefore there is a subspace $V \leq V_t$ of codimension (relative to V_t) $\ll \mathcal{L}(\alpha)^8$ such that there exists some $x \in V_t$ with

$$\frac{|(A_t - x) \cap V|}{|V|} = 1_{A_t} * \mu_V(x) = \|1_{A_t} * \mu_V\|_\infty \geq (1 + 1/64)\alpha_t,$$

which contradicts the maximality of t , letting $V_{t+1} = V$ and $A_{t+1} = (A_t - x) \cap V_t$.

Therefore

$$|V_t| \langle \mu_{A_t} * \mu_{A_t}, \mu_{2 \cdot A_t} \rangle \geq 1/2.$$

By Lemma 5.7 the left-hand side is equal to $|V_t| / |A_t|^2$, and therefore

$$\alpha^2 \leq \alpha_t^2 \leq 2/|V_t|.$$

By the codimension bound the right-hand side is at most

$$2q^{O(\mathcal{L}(\alpha)^9) - n}.$$

If $\alpha^2 \leq 2q^{-n/2}$ we are done, otherwise we deduce that $\mathcal{L}(\alpha)^9 \gg n$ as required. \square

Chapter 6

Bohr sets

Definition 6.1 (Bohr sets). *Let $\nu : \widehat{G} \rightarrow \mathbb{R}$. The corresponding Bohr set is defined to be*

$$\text{Bohr}(\nu) = \{x \in G : |1 - \gamma(x)| \leq \nu(\gamma) \text{ for all } \gamma \in \Gamma\}.$$

The rank of ν , denoted by $\text{rk}(\nu)$, is defined to be the size of the set of those $\gamma \in \widehat{G}$ such that $\nu(\gamma) < 2$.

(Basic API facts: Bohr sets are symmetric and contain 0. Also that, without loss of generality, we can assume ν takes only values in $\mathbb{R}_{\geq 0}$ - I think it might be easier to have the definition allow arbitrary real values, and then switch to non-negative only in proofs where convenient. Or could have the definition only allow non-negative valued functions in the first place.)

Lemma 6.2.

If $\rho \in (0, 1)$ and $\nu : \widehat{G} \rightarrow \mathbb{R}$ then

$$|\text{Bohr}(\rho \cdot \nu)| \geq (\rho/4)^{\text{rk}(\nu)} |\text{Bohr}(\nu)|.$$

Proof. There are at most $\lceil 4/\rho \rceil$ many z_i such that if $|1 - w| \leq \nu(\gamma)$ then $|z_i - w| \leq \rho\nu(\gamma)/2$ for some i . Let $\Gamma = \{\gamma : \nu(\gamma) < 2\}$ and define a function $f : \text{Bohr}(\nu) \rightarrow \lceil 2/\rho \rceil^{\text{rk}(\nu)}$ where for $\gamma \in \Gamma$ we assign the γ -coordinate of $f(x)$ as whichever j has $|z_j - \gamma(x)| \leq \rho\nu(\gamma)/2$.

By the pigeonhole principle there must exist some (j_1, \dots, j_d) such that $f^{-1}(j_1, \dots, j_d)$ has size at least $(\lceil 2/\rho \rceil)^{-\text{rk}(\nu)} |\text{Bohr}(\nu)|$. Call this set B' . It must be non-empty, so fix some $x \in B'$. We claim that $B' - x \subseteq |\text{Bohr}(\rho \cdot \nu)|$, which completes the proof.

Suppose that $z = x + y$ with $x, y \in B'$, and fix some $\gamma \in \Gamma$. By assumption there is some $z_j \in \mathbb{C}$ such that $|z_j - \gamma(x)| \leq \rho\nu(\gamma)/2$ and $|z_j - \gamma(y)| \leq \rho\nu(\gamma)/2$. Then by the triangle inequality,

$$|1 - \gamma(y - x)| = |\gamma(x) - \gamma(y)| \leq \rho\nu(\gamma)$$

and so $z = y - x \in \text{Bohr}(\rho \cdot \nu)$. □

Definition 6.3 (Regularity). *We say $\nu : \widehat{G} \rightarrow \mathbb{R}$ is regular if, with $d = \text{rk}(\nu)$, for all $\kappa \in \mathbb{R}$ with $|\kappa| \leq 1/100d$ we have*

$$(1 - 100d|\kappa|) \leq \frac{|\text{Bohr}((1 + \kappa)\nu)|}{|\text{Bohr}(\nu)|} \leq (1 + 100d|\kappa|)$$

Lemma 6.4. *For any $\nu : \widehat{G} \rightarrow \mathbb{R}$ there exists $\rho \in [\frac{1}{2}, 1]$ such that $\rho \cdot \nu$ is regular.*

Proof. To do. □

Lemma 6.5. *If B is a regular Bohr set of rank d and $\mu : G \rightarrow \mathbb{R}_{\geq 0}$ is supported on B_ρ , with $\rho \in (0, 1)$, then*

$$\|\mu_B * \mu - \mu_B\|_1 \ll \rho d \|\mu\|_1.$$

Proof. To do. □

Lemma 6.6. *There is a constant $c > 0$ such that the following holds. Let B be a regular Bohr set of rank d and $L \geq 1$ be any integer. If $\nu : G \rightarrow \mathbb{R}_{\geq 0}$ is supported on LB_ρ , where $\rho \leq c/Ld$, and $\|\nu\|_1 = 1$, then*

$$\mu_B \leq 2\mu_{B_{1+L\rho}} * \nu.$$

Proof. To do. □

Lemma 6.7. *There is a constant $c > 0$ such that the following holds. Let B be a regular Bohr set of rank d , suppose $A \subseteq B$ has density α , let $\epsilon > 0$, and suppose $B', B'' \subseteq B_\rho$ where $\rho \leq c\alpha\epsilon/d$. Then either*

1. *there is some translate A' of A such that $|A' \cap B'| \geq (1 - \epsilon)\alpha|B'|$ and $|A' \cap B''| \geq (1 - \epsilon)\alpha|B''|$, or*
2. *$\|1_A * \mu_{B'}\|_\infty \geq (1 + \epsilon/2)\alpha$, or*
3. *$\|1_A * \mu_{B''}\|_\infty \geq (1 + \epsilon/2)\alpha$.*

Proof. To do. □

Chapter 7

The integer case

Theorem 7.1. *There is a constant $c > 0$ such that the following holds. Let $\epsilon > 0$ and $B, B' \subseteq G$ be regular Bohr sets of rank d . Suppose that $A_1 \subseteq B$ with density α_1 and A_2 is such that there exists x with $A_2 \subseteq B' - x$ with density α_2 . Let S be any set with $|S| \leq 2|B|$. There is a regular Bohr set $B'' \subseteq B'$ of rank at most*

$$d + O_\epsilon(\mathcal{L}\alpha_1^3\mathcal{L}\alpha_2)$$

and size

$$|B''| \geq \exp(-O_\epsilon(d\mathcal{L}\alpha_1\alpha_2/d + \mathcal{L}\alpha_1^3\mathcal{L}\alpha_2\mathcal{L}\alpha_1\alpha_2/d))|B'|$$

such that

$$|\langle \mu_{B'} * \mu_{A_1} \circ \mu_{A_2}, 1_S \rangle - \langle \mu_{A_1} \circ \mu_{A_2}, 1_S \rangle| \leq \epsilon.$$

Proof. To do. □

Proposition 7.2. *There is a constant $c > 0$ such that the following holds. Let $\epsilon > 0$ and $p \geq 2$ be an integer. Let $B \subseteq G$ be a regular Bohr set and $A \subseteq B$ with relative density α . Let $\nu : G \rightarrow \mathbb{R}_{\geq 0}$ be supported on B_ρ , where $\rho \leq c\epsilon\alpha / \text{rank}(B)$, such that $\|\nu\|_1 = 1$ and $\hat{\nu} \geq 0$. If*

$$\|(\mu_A - \mu_B) \circ (\mu_A - \mu_B)\|_{p(\nu)} \geq \epsilon \mu(B)^{-1},$$

then there exists $p' \ll_\epsilon p$ such that

$$\|\mu_A \circ \mu_A\|_{p'(\nu)} \geq (1 + \frac{1}{4}\epsilon) \mu(B)^{-1}.$$

Proof. To do. □

Proposition 7.3. *There is a constant $c > 0$ such that the following holds. Let $p \geq 2$ be an even integer. Let $f : G \rightarrow \mathbb{R}$, let $B \subseteq G$ and $B', B'' \subseteq B_{c/\text{rank}(B)}$ all be regular Bohr sets. Then*

$$\|f \circ f\|_{p(\mu_{B'} \circ \mu_{B'} * \mu_{B''} \circ \mu_{B''})} \geq \frac{1}{2} \|f * f\|_{p(\mu_B)}.$$

Proof. To do, □

Proposition 7.4. *There is a constant $c > 0$ such that the following holds. Let $\epsilon > 0$. Let $B \subseteq G$ be a regular Bohr set and $A \subseteq B$ with relative density α , and let $B' \subseteq B_{c\epsilon\alpha/\text{rank}(B)}$ be a regular Bohr set and $C \subseteq B'$ with relative density γ . Either*

1. $|\langle \mu_A * \mu_A, \mu_C \rangle - \mu(B)^{-1}| \leq \epsilon \mu(B)^{-1}$ or
2. there is some $p \ll \mathcal{L}\gamma$ such that $\|(\mu_A - \mu_B) * (\mu_A - \mu_B)\|_{p(\mu_{B'})} \geq \frac{1}{2}\epsilon \mu(B)^{-1}$.

Proof. To do. □

Proposition 7.5. *There is a constant $c > 0$ such that the following holds. Let $\epsilon > 0$ and $p, k \geq 1$ be integers such that $(k, |G|) = 1$. Let $B, B', B'' \subseteq G$ be regular Bohr sets of rank d such that $B'' \subseteq B'_{c/d}$ and $A \subseteq B$ with relative density α . If*

$$\|\mu_A \circ \mu_A\|_{p(\mu_{k \cdot B'} \circ \mu_{k \cdot B'} * \mu_{k \cdot B''} \circ \mu_{k \cdot B''})} \geq (1 + \epsilon) \mu(B)^{-1},$$

then there is a regular Bohr set $B''' \subseteq B''$ of rank at most

$$\text{rank}(B''') \leq d + O_\epsilon(\mathcal{L}\alpha^4 p^4)$$

and size

$$|B'''| \geq \exp(-O_\epsilon(dp\mathcal{L}\alpha/d + \mathcal{L}\alpha^5 p^5)) |B''|$$

such that

$$\|\mu_{B'''} * \mu_A\|_\infty \geq (1 + c\epsilon) \mu(B)^{-1}.$$

Proof. To do. □

Theorem 7.6. *There is a constant $c > 0$ such that the following holds. Let $\epsilon, \delta \in (0, 1)$ and $p, k \geq 1$ be integers such that $(k, |G|) = 1$. For any $A \subseteq G$ with density α there is a regular Bohr set B with*

$$d = \text{rank}(B) = O_\epsilon(\mathcal{L}\alpha^5 p^4) \quad \text{and} \quad |B| \geq \exp(-O_{\epsilon, \delta}(\mathcal{L}\alpha^6 p^5 \mathcal{L}\alpha/p)) |G|$$

and some $A' \subseteq (A - x) \cap B$ for some $x \in G$ such that

1. $|A'| \geq (1 - \epsilon)\alpha |B|$,
2. $|A' \cap B'| \geq (1 - \epsilon)\alpha |B'|$, where $B' = B_\rho$ is a regular Bohr set with $\rho \in (\frac{1}{2}, 1) \cdot c\delta\alpha/d$, and
3. $\|\mu_{A'} \circ \mu_{A'}\|_{p(\mu_{k \cdot B''} \circ \mu_{k \cdot B''} * \mu_{k \cdot B'''} \circ \mu_{k \cdot B'''})} < (1 + \epsilon) \mu(B)^{-1}$,

for any regular Bohr sets $B'' = B'_{\rho'}$ and $B''' = B''_{\rho''}$ satisfying $\rho', \rho'' \in (\frac{1}{2}, 1) \cdot c\delta\alpha/d$.

Proof. To do. □

Theorem 7.7. *There is a constant $c > 0$ such that the following holds. Let $\delta, \epsilon \in (0, 1)$, let $p \geq 1$ and let k be a positive integer such that $(k, |G|) = 1$. There is a constant $C = C(\epsilon, \delta, k) > 0$ such that the following holds.*

For any finite abelian group G and any subset $A \subseteq G$ with $|A| = \alpha |G|$ there exists a regular Bohr set B with

$$\text{rank}(B) \leq Cp^4 \log(2/\alpha)^5$$

and

$$|B| \geq \exp(-Cp^5 \log(2p/\alpha) \log(2/\alpha)^6) |G|$$

and $A' \subseteq (A - x) \cap B$ for some $x \in G$ such that

1. $|A'| \geq (1 - \epsilon)\alpha |B|$,
2. $|A' \cap B'| \geq (1 - \epsilon)\alpha |B'|$, where $B' = B_\rho$ is a regular Bohr set with $\rho \in (\frac{1}{2}, 1) \cdot c\delta\alpha/dk$, and
3. $\|(\mu_{A'} - \mu_B) * (\mu_{A'} - \mu_B)\|_{p(\mu_{k \cdot B'})} \leq \epsilon \frac{|G|}{|B|}$.

Proof. To do. □

Theorem 7.8. *If $A \subseteq \{1, \dots, N\}$ has size $|A| = \alpha N$, then A contains at least*

$$\exp(-O(\mathcal{L}\alpha^{12}))N^2$$

many three-term arithmetic progressions.

Proof. To do. □

Theorem 7.9 (Integer case). *If $A \subseteq \{1, \dots, N\}$ contains no non-trivial three-term arithmetic progressions then*

$$|A| \leq \frac{N}{\exp(-c(\log N)^{1/12})}$$

Proof. To do. □