Toric

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The presentation is inspired by [1], but we do not aim to follow it very closely. For example, the chapters below do not match those of [1].

Chapter 1

Category theory

1.1 Over category

Proposition 1.1.1 (Sliced adjoint functors). *If $a : F \vdash G$ is an adjunction between $F : C \to D$ and $G : D \to C$ and $X : C$, then there is an adjunction between $F/X : C/X \to D/F(X)$ and $G/X : D/F(X) \to C/X$.*

Proof. See https://ncatlab.org/nlab/show/sliced+adjoint+functors+−+section. □

Proposition 1.1.2 (Limit-preserving functors lift to over categories). *Let J be a shape (i.e. a category). Let $\widetilde{J}$ denote the category which is the same as J, but has an extra object $*$ which is terminal. If $F : C \to D$ is a functor preserving limits of shape $\widetilde{J}$, then the obvious functor $C/X \to D/F(X)$ preserves limits of shape J.*

Proof. Extend a functor $K : J \to C/X$ to a functor $\widetilde{K} : \widetilde{J} \to C$, by letting $\widetilde{K}(*) = X$. □

Proposition 1.1.3 (Essential image of a sliced functor). *If $F : C \to D$ is a full functor between cartesian-monoidal categories, then $F/X : C/X \hom D/F(X)$ has the same essential image as F.*

Proof. Transfer all diagrams. □

1.2 Objects

1.2.1 Group objects

Proposition 1.2.1 (Fully faithful product-preserving functors lift to monoid/group objects). *If a finite-products-preserving functor $F : C \to D$ is fully faithful, then so is $\mathrm{Grp}(F) : \mathrm{Grp}\, C \to \mathrm{Grp}\, D$.*

Proof. Faithfulness is immediate.

For fullness, assume $f : F(G) \to F(H)$ is a morphism. By fullness of F, find $g : G \to H$ such that $F(g) = f$. g is a morphism because we can pull back each diagram from D to C along F which is faithful. □

Proposition 1.2.2 (Equivalences lift to monoid/group object categories). *If $e : C \backsimeq D$ is an equivalence of cartesian-monoidal categories, then $\mathrm{Grp}(e) : \mathrm{Grp}(C) \backsimeq \mathrm{Grp}(D)$ too is an equivalence of categories.*

Proof. Transfer all diagrams. ☐

Proposition 1.2.3 (Essential image of a functor on monoid/group objects). *If $F : C \to D$ is a fully faithful functor between cartesian-monoidal categories, then* $\mathrm{Grp}(F) : \mathrm{Grp}(C) \hom \mathrm{Grp}(D)$ *has the same essential image as F.*

Proof.
Transfer all diagrams. ☐

Lemma 1.2.4 (A fully faithful functor product-preserving is a group isomorphism on hom sets).

If $F : C \to D$ is a fully faithful functor between cartesian-monoidal categories and $X, G \in C$ are an object and a group object respectively, then $\mathrm{Grp}(F) : (X \hom G) \hom \cong (F(X) \hom F(G))$ *is a group isomorphism.*

Proof. Toddlers and streets. ☐

1.2.2 Module objects

Proposition 1.2.5 (Pulling back a module object). *Let M, N be two monoid objects in a monoidal category C. Let $f : M \to N$ be a monoid morphism. If X is a N-module object, then it is also a M-module object.*

Proof. Define the multiplication $\mu_M : M \otimes X \to X$ as $f \otimes \mathbb{1} \gg \mu_N$. All proofs follow easily. ☐

Chapter 2

Algebra

2.1 Tensor Product

Lemma 2.1.1 (The tensor product of linearly independent families)**.** *Let R be a domain and M, N two R-semimodules. If f and g are linearly independent families of points in M and N, then $(i, j) \mapsto fi \otimes gj$ is a linearly independent family of points in $M \otimes N$.*

Proof. We will prove the equivalent statement:

Let P, Q be two free R modules, $f : P \to M$ and $g : Q \to N$ be two R-linear injective maps. Then $f \otimes g : P \otimes_R Q \to M \otimes_R N$ is injective.

Let K be the field of fractions of R.

The map

$$P \otimes_R Q \to (K \otimes_R P) \otimes_R (K \otimes_R Q) = (K \otimes_R P) \otimes_K (K \otimes_R Q)$$

is injective because $R \to K$ is injective and all the modules involved are flat. The map

$$(K \otimes_R P) \otimes_K (K \otimes_R Q) \to (K \otimes_R M) \otimes_K (K \otimes_R N)$$

is injective because all the modules involved are K-flat (as K is a field).

$P \otimes_R Q \to M \otimes_R N$ is now a factor of the composition of the two injections above, and is thus is injective. □

2.2 Affine Monoids

Lemma 2.2.1 (Multivariate Laurent polynomials are an integral domain)**.** *Multivariate Laurent polynomials over an integral domain are an integral domain.*

Proof. Come on. □

Definition 2.2.2 (Affine monoid)**.** An *affine monoid* is a finitely generated commutative monoid which is:

- cancellative: if $a + c = b + c$ then $a = b$, and
- torsion-free: if $na = nb$ then $a = b$ (for $n \geq 1$).

Proposition 2.2.3 (Embedding an affine monoid inside a lattice)**.**
If M is an affine monoid, then M can be embedded inside $\mathbb{Z}^n$ for some n.

Proof. Embed M inside its Grothendieck group G. Prove that G is finitely generated free. □

Proposition 2.2.4 (Affine monoid algebras are domains)**.**
If R is an integral domain M is an affine monoid, then $R[M]$ is an integral domain and is a finitely generated R-algebra.

Proof.
$i : R[M] \hookrightarrow R[\mathbb{Z}M]$ injects into an integral domain so is an integral domain. It's finitely generated by χ^{a_i} where $\mathcal{A} = \{a_1, \ldots, a_s\}$ is a finite generating set for M. □

Definition 2.2.5 (Irreducible element)**.** An element x of a monoid M is *irreducible* if $x = y + z$ implies $y = 0$ or $z = 0$.

Proposition 2.2.6 (Irreducible elements lie in all sets generating a salient monoid)**.**
If M is a monoid with a single unit, and S is a set generating M, then S contains all irreducible elements of M.

Proof. Assume p is an irreducible element. Since S generates M, write

$$p = \sum_i a_i$$

where the a_i are finitely many elements (not necessarily distinct) elements of S. Since p is irreducible, we must have

$$p = a_i \in S$$

for some i. □

Proposition 2.2.7 (A salient finitely generated monoid has finitely many irreducible elements)**.**

If M is a finitely generated monoid with a single unit, then only finitely many elements of M are irreducible.

Proof.
Let S be a finite set generating M. Write I the set of irreducible elements. By Proposition 2.2.6, $I \subseteq S$. Hence I is finite. □

Proposition 2.2.8 (A salient finitely generated cancellative monoid is generated by its irreducible elements)**.**
If M is a finitely generated cancellative monoid with a single unit, then M is generated by its irreducible elements.

Proof. We do not follow the proof from [1].
Let S be a finite minimal generating set and assume for contradiction that $r \in S$ is reducible, say $r = a + b$ where a, b are non-units. Write

$$a = \sum_{s \in S} m_s s, b = \sum_{s \in S} n_s s$$

for some $m_s, n_s \in \mathbb{N}$, so that

$$r = \sum_{s \in S} (m_s + n_s) s.$$

We distinguish three cases

- $m_r + n_r = 0$. Then

$$r = \sum_{s \in S \setminus \{r\}} (m_s + n_s)s \in \langle S \setminus \{r\} \rangle$$

contradicting the minimality of S.

- $m_r + n_r = 1$. Then

$$0 = \sum_{s \in S \setminus \{r\}} (m_s + n_s)s \qquad \implies \forall s \in S \setminus \{r\}, m_s s = n_s s = 0$$

Furthermore, either $m_r = 0$ or $n_r = 0$, so $a = 0$ or $b = 0$, contradicting the fact that a and b are non-units.

- $m_r + n_r \geq 2$. Then

$$0 = r + \sum_{s \in S \setminus \{r\}} (m_s + n_s)s$$

and $r = 0$, contradicting the minimality of S once again.

□

2.3 Hopf algebras

2.3.1 Ideals and quotients

Definition 2.3.1 (Coideal)**.** Let R be a commutative ring and (C, Δ, ε) be a coalgebra over R. An R-submodule I of C is a *coideal* of C if $\Delta(I) \subseteq I \bar{\otimes}_R C + C \bar{\otimes}_R I$ and $\varepsilon(I) = 0$, where $\bar{\cdot}$ denotes the image in $C \otimes_R C$.

Proposition 2.3.2 (Quotient coalgebra)**.**
If C is a coalgebra over R and I is a coideal, the quotient C/I is equipped with a canonical R-coalgebra structure.

Proof. Straightforward. □

Proposition 2.3.3 (Quotient coalgebra map)**.**
If C is a coalgebra over R and I is a coideal, the quotient map $C \to C/I$ is a coalgebra homomorphism.

Proof. Straightforward. □

Definition 2.3.4 (Bialgebra ideal)**.**
Let B be a bialgebra over a commutative ring R. A *bialgebra ideal* I is an ideal which is also a coideal.

Proposition 2.3.5 (Quotient bialgebra)**.**
If B is a bialgebra over R and I is a bialgebra ideal, the quotient B/I is equipped with a canonical R-bialgebra structure.

Proof.
Straightforward. □

Proposition 2.3.6 (Quotient bialgebra map)**.**
If B is a bialgebra over R and I is a bialgebra ideal, the quotient map $B \to B/I$ is a bialgebra homomorphism.

Proof.
Straightforward. □

Definition 2.3.7 (Hopf ideal)**.**
Let A be a Hopf algebra over a commutative ring R. A *Hopf ideal* I is a bialgebra ideal such that $S(I) = I$.

Proposition 2.3.8 (Quotient Hopf algebra)**.**
If A is a Hopf algebra over R and I is a Hopf ideal, the quotient A/I is equipped with a canonical Hopf algebra structure over R.

Proof.
Straightforward. □

Proposition 2.3.9 (Quotient Hopf algebra map)**.**
If A is a Hopf algebra over R and I is a Hopf ideal, the quotient map $A \to A/I$ is a Hopf algebra homomorphism.

Proof.
Follows immediately from Proposition 2.3.6. □

2.3.2 Group algebras

Proposition 2.3.10 (Freeness of group algebras under an injective hom)**.** *Let R be a commutative ring. Let G, H be abelian groups and $f : G \to H$ an injective group hom. Then $R[H]$ is a free $R[G]$-module.*

Proof. Pick a section $\sigma : H/f(G) \to H$ and the unique map $\varphi : H \to G$ such that $h = \sigma(hf(G))f(\varphi(h))$. We claim that $R[H]$ is isomorphic to $R[G]^{\oplus H/f(G)}$, from which the result follows, as such:

$$\begin{aligned} R[G]^{\oplus H/f(G)} &\simeq R[H] \\ \varphi(h)e_{hf(G)} &\mapsto hge_x \qquad\qquad \hookleftarrow \sigma(x)f(g) \end{aligned}$$

Those two functions are clearly inverse to each other, and the forward map is clearly $R[G]$-linear. □

Proposition 2.3.11 (The kernel of a map on direct sums)**.** *Let G be an abelian group generated by a set S. Let A, B be arbitrary indexing types and $f : A \to B$ a function. Write $f^{\oplus} : G^{\oplus A} \to G^{\oplus B}$ the pushforward. Then*

$$\ker f^{\oplus} = \operatorname{span}\{gX_1^a - gX_2^a | g \in S, a_1, a_2 \in A, f(a_1) = f(a_2)\}.$$

Proof. Write $I = \operatorname{span}\{gX_1^a - gX_2^a | g \in G, a_1, a_2 \in A, f(a_1) = f(a_2)\}$ for brevity.
Note that we can assume WLOG that f is surjective. Write $\sigma : B \to A$ a section of f.
Let's prove by induction on $x \in G^{\oplus}A$ that $\sigma^{\oplus}(f^{\oplus}(x)) \equiv x \mod I$:

- $x = 0$: $\sigma^{\oplus}(f^{\oplus}(0)) = 0$
- $x = gX^a$: $\sigma^{\oplus}(f^{\oplus}(gX^a)) = gX^{\sigma(f(a))} \equiv gX^a \mod I$ as S generates

- $x+y$: Assume the induction hypothesis for x and y. Then

$$\sigma^{\oplus}(f^{\oplus}(x+y)) = \sigma^{\oplus}(f^{\oplus}(x)) + \sigma^{\oplus}(f^{\oplus}(y)) \equiv x+y \mod I$$

Now, for any $x \in G^{\oplus}A$,

$$x \in \ker f^{\oplus} \iff f^{\oplus}(x) = 0 \iff \sigma^{\oplus}(f^{\oplus}(x)) \equiv 0 \mod I \iff x \equiv 0 \mod I$$

and we are done. □

Proposition 2.3.12 (Localising a monoid algebra)**.** *Let R be a commutative ring. Let M be a commutative monoid and M' be its localization at some $S \subseteq M$. Then $R[M']$ is the localization of $R[M]$ at* $\operatorname{span}\{X^s | s \in S\}$.

Proof. Straightforward. □

2.3.3 Group-like elements

Definition 2.3.13 (Group-like elements)**.** An element a of a coalgebra A is *group-like* if $\eta(a) = 1$ and $\Delta(a) = a \otimes a$, where η is the counit and Δ is the comultiplication map.

We write $\operatorname{GrpLike} A$ for the set of group-like elements of A.

Proposition 2.3.14 (Group-like elements form a group)**.**

Group-like elements $\operatorname{GrpLike} A$ *of a bialgebra A form a monoid.*

Group-like elements $\operatorname{GrpLike} A$ *of a Hopf algebra A form a group.*

Proof. Check that group-like elements are closed under unit, multiplication and inverses. □

Lemma 2.3.15 (Bialgebra homs preserve group-like elements)**.**

Let $f : A \to B$ be a bi-algebra hom. If $a \in A$ is group-like, then $f(a)$ is group-like too.

Proof. a is a unit, so $f(a)$ is a unit too. Then

$$f(a) \otimes f(a) = (f \otimes f)(\Delta_A(a)) = \Delta_B(f(a))$$

so $f(a)$ is group-like. □

Lemma 2.3.16.

If R is a commutative semiring, A is a Hopf algebra over R and G is a group, then every element of the image of G in $A[G]$ is group-like.

Proof. This is an easy check. □

Lemma 2.3.17.

If R is a commutative semiring, A is a Hopf algebra over R and G is a group, then the group-like elements in $A[G]$ span $A[G]$ as an A-module.

Proof.

This follows immediately from 2.3.16. □

Lemma 2.3.18 (Independence of group-like elements)**.**

The group-like elements in a bialgebra A over a domain are linearly independent.

Proof.

Let's prove that any finite set s of group-like elements is linearly independent, by induction on s.

$\emptyset$ is clearly linearly independent.

Assume now that the finite set s of group-like elements is linearly independent, that $a \notin s$ is group-like, and let's show that $s \cup \{a\}$ is linearly independent too.

Assume there is some $c : A \to R$ such that $\sum_{x \in s} c_x x = c_a a$. Since a and all elements of s are group-like, we compute

$$\begin{aligned}\sum_{x,y \in s} c_x c_y x \otimes y &= c_a^2 a \otimes a \\ &= c_a^2 \Delta(a) \\ &= c_a \Delta \left(\sum_{x \in s} c_x x \right) \\ &= \sum_{x \in s} c_a c_x \Delta(x) \\ &= \sum_{x \in s} c_a c_x x \otimes x\end{aligned}$$

By Lemma 2.1.1, the $x \otimes y$ are linearly independent and therefore $c_x^2 = c_a c_x$ and $c_x c_y = 0$ if $x \neq y$.

If $c_x = 0$ for all $x \in s$, then we are clearly done. Else find $x \in s$ such that $c_x \neq 0$. From the above two equations, we get that $c_x = c_a$ and $c_y = 0$ for all $y \in s, y \neq x$. Therefore

$$c_x x = \sum_{y \in s} c_y y = c_a a = c_x a$$

and $x = a$. Contradiction. □

Lemma 2.3.19 (Group-like elements in a group algebra)**.**

Let R be a domain. The group-like elements of $R[M]$ are exactly the image of M.

Proof.

See Lemma 12.4 in [2]. □

Proposition 2.3.20 (Galois connection between group algebra and group-like elements)**.**

Let R be a domain, G a commutative group and A a R-bialgebra. Then bialgebra homs $R[G] \to A$ are in bijection with group homs $G \to \operatorname{GrpLike} A$.

Proof.

If $f : G \to \operatorname{GrpLike} A$ is a group hom, then we get

$$R[G] \to Ag \mapsto f(g)$$

This is clearly an algebra hom, so for it to be a bialgebra hom we only need to check comultiplication is preserved. We only need to check this on $g \in G$, in which case

$$(f \otimes f)(\Delta(g)) = (f \otimes f)(g \otimes g) = f(g) \otimes f(g) = \Delta(f(g))$$

since $f(g) \in \operatorname{GrpLike} A$.

If $f : R[G] \to A$ is a bialgebra hom, then it restricts to a group hom $\operatorname{GrpLike} R[G] \to \operatorname{GrpLike} A$ by Proposition 2.3.15. Now use that $\operatorname{GrpLike} R[G] \cong G$ from Proposition 2.3.19. □

Proposition 2.3.21 (Quotients by binomial ideals)**.**

Let A be a Hopf algebra, H be a subgroup of $\operatorname{GrpLike} A$ *and*

$$I = \langle h_1 - h_2 : h_1, h_2 \in H \rangle$$

be an ideal. Then I is a Hopf ideal.

Proof. It suffices to check the conditions of a Hopf ideal on generators.

For the comultiplication condition:

$$\begin{aligned}\Delta(h_1 - h_2) &= \Delta(h_1) - \Delta(h_2) \\ &= h_1 \otimes h_1 - h_2 \otimes h_2 \\ &= h_1 \otimes h_1 - h_1 \otimes h_2 + h_1 \otimes h_2 - h_2 \otimes h_2 \\ &= h_1 \otimes (h_1 - h_2) + (h_1 - h_2) \otimes h_2 \in A \mathbin{\bar{\otimes}} I + I \mathbin{\bar{\otimes}} A.\end{aligned}$$

For the counit condition:

$$\varepsilon(h_1 - h_2) = \varepsilon(h_1) - \varepsilon(h_2) = 1 - 1 = 0.$$

Finally, for the antipode condition:

$$S(h_1 - h_2) = S(h_1) - S(h_2) = h_1^{-1} - h_2^{-1} \in I.$$

□

2.3.4 Diagonalizable bialgebras

Definition 2.3.22 (Diagonalizable bialgebras)**.** A bialgebra is called diagonalizable if it is isomorphic to a group algebra.

Lemma 2.3.23.

A diagonalizable bialgebra is spanned by its group-like elements.

Proof.

This is true for a group algebra by 2.3.17, and the property of being spanned by its group-like elements is preserves by isomorphisms of bialgebras. □

Proposition 2.3.24.

Let A be a bialgebra over a domain R, let G be a subgroup of $\operatorname{GrpLike}(A)$ *(which is a monoid by 2.3.14). If A is generated by G, then the unique bialgebra morphism from $R[G]$ to A sending each element of G to itself is bijective.*

Proof.

This morphism is injective by the linear independence of group-like elements (2.3.18), and surjective by assumption. □

Proposition 2.3.25 (Quotient of a diagonalisable bialgebra is diagonalisable)**.**

Let R be a domain, G a commutative group, A a R-bialgebra and $f : R[G] \to A$ a surjective bialgebra hom. Then $R[f(G)] \cong A$ as bialgebras.

Proof.

Note that $R[G] \xrightarrow{f} A$ factors as $R[G] \xrightarrow{f} R[f(G)] \xrightarrow{\phi} A$, where $f(G)$ is a group by Proposition 2.3.14.

Since $R[G] \xrightarrow{f} A$ is surjective, so is $R[f(G)] \xrightarrow{\phi} A$. Therefore Proposition 2.3.24 applies to $f(G)$ inside A, and we get $R[f(G)] \cong A$. □

Corollary 2.3.26.

A bialgebra over a domain is diagonalizable if and only if it is spanned by its group-like elements.

Proof.

We know that a diagonalizable bialgebra is spanned by its group-like elements by 2.3.23, and that a bialgebra over a domain that is spanned by its group-like elements is diagonalizable by 2.3.25 (and by the fact that a bijective morphism of bialgebras is an isomorphism). □

Proposition 2.3.24 and Corollary 4.2.4 are false over a general commutative ring. Indeed, let R be a commutative ring and let G be a group. Then the group-like elements of $R[G]$ correspond to locally constant maps from $SpecR$ to G (with the discrete topology), hence they are of the form $e_1g_1 + \cdots + e_rg_r$, with the g_i in G and $e_1, \ldots, e_r$ a family of pairwise orthogonal idempotent elements of R that sum to 1. So $R[G]$ is not isomorphic to the group algebra over its group-like elements unless $SpecR$ is connected. As for the corollary, a bialgebra of the form $R_1[G_1] \times \cdots \times R_n[G_n]$, seen as a bialgebra over $R_1 \times \cdots \times R_n$, is generated by its group-like elements but not diagonalizable.

2.3.5 The group algebra functor

Proposition 2.3.27 (The antipode is a antihomomorphism)**.** *If A is a R-Hopf algebra, then the antipode map $s : A \to A$ is anti-commutative, ie $s(a * b) = s(b) * s(a)$. If further A is commutative, then $s(a * b) = s(a) * s(b)$.*

Proof. Any standard reference will have a proof. □

Proposition 2.3.28 (Bialgebras are comonoid objects in the category of algebras)**.** *The category of R-bialgebras is equivalent to comonoid objects in the category of R-algebras.*

Proof. Turn the arrows around. □

Proposition 2.3.29 (Hopf algebras are cogroup objects in the category of algebras)**.** *The category of R-Hopf algebras is equivalent to cogroup objects in the category of R-algebras.*

Proof.

Turn the arrows around. Most of the diagrams have been turned around in Proposition 2.3.28 already. □

Definition 2.3.30 (The group algebra functor)**.** For a commutative ring R, we have a functor $G \rightsquigarrow R[G] : \mathrm{Grp} \to \mathrm{Hopf}_R$.

Proposition 2.3.31 (The group algebra functor is fully faithful)**.**

Let R be a domain. The functor $G \rightsquigarrow R[G]$ from the category of groups to the category of Hopf algebras over R is fully faithful.

Proof.

The functor is clearly faithful. Now for the full part, if $f : R[G] \to R[H]$ is a Hopf algebra hom, then we get a series of maps

$$G \simeq \text{ group-like elements of } R[G] \to \text{ group-like elements of } R[H] \simeq H$$

and each map separately is clearly multiplicative. □

Chapter 3

Convex geometry

3.1 Cones

3.1.1 Convex Polyhedral Cones

Fix a pair of dual real vector spaces M and N.

Definition 3.1.1 (Convex cone generated by a set)**.** For a set $S \subseteq N$, the **cone generated by** S, aka **cone hull of** S, is

$$\mathrm{Cone}(S) := \left\{ \sum_{u \in S} \lambda_u u | \lambda_u \geq 0 \right\}$$

Definition 3.1.2 (Convex polyhedral cone)**.**
A **polyhedral cone** is a set that can be written as $\mathrm{Cone}(S)$ for some finite set S.

Definition 3.1.3 (Convex hull)**.** For a set $S \subseteq N$, the **convex hull of** S is

$$\mathrm{Conv}(S) := \left\{ \sum_{u \in S} \lambda_u | \lambda_u \geq 0, \sum_{u} \lambda_u = 1 \right\}$$

Definition 3.1.4 (Polytope)**.**
A **polytope** is a set that can be written as $\mathrm{Conv}(S)$ for some finite set S.

3.1.2 Dual Cones and Faces

Definition 3.1.5 (Dual cone)**.**
Given a polyhedral cone $\sigma \subseteq N$, its **dual cone** is defined by

$$\sigma^\vee = \{ m \in M | \forall u \in \sigma, \langle m, u \rangle \geq 0 \}$$

.

Proposition 3.1.6 (Dual of a polyhedral cone)**.**
If σ is polyhedral, then its dual $\sigma^\vee$ is polyhedral too.

Proof. Classic, use Fourier-Motzkin eliminiation. □

Proposition 3.1.7 (Dual cone of a sumset).

If σ_1, σ_2 are two cones, then

$$(\sigma_1 + \sigma_2)^\vee = \sigma_1^\vee \cap \sigma_2^\vee.$$

Proof. Classic. See [3] maybe. □

Proposition 3.1.8 (Double dual of a polyhedral cone).

If σ is polyhedral, then $\sigma^{\vee\vee} = \sigma$.

Proof. Classic. See [3] maybe. □

Given $m \neq 0$ in M, we get the hyperplane

$$H_m = \{u \in N | \langle m, u\rangle = 0\} \subseteq N$$

and the closed half-space

$$H_m^+ = \{u \in N | \langle m, u\rangle \geq 0\} \subseteq N.$$

Definition 3.1.9 (Face of a cone). If σ is a cone, then a subset of σ is a **face** iff it is the intersection of σ with some halfspace. We write this $\tau \preceq \sigma$. If furthermore $\tau \neq \sigma$, we call τ a proper face and write $\tau \prec \sigma$.

Definition 3.1.10 (Edge of a cone).

A dimension 1 face of a cone is called an *edge*.

Definition 3.1.11 (Facet of a cone).

A codimension 1 face of a cone is called a *facet*.

Lemma 3.1.12 (Face of a polyhedral cone).

If σ is a polyhedral cone, then every face of σ is a polyhedral cone.

Lemma 3.1.13 (Intersection of faces).

If σ is a polyhedral cone, then the intersection of two faces of σ is a face of σ.

Proof. Classic. See [3] maybe. □

Lemma 3.1.14 (Face of a face).

A face of a face of a polyhedral cone σ is again a face of σ.

Proof. Classic. See [3] maybe. □

Lemma 3.1.15.

Let τ be a face of a polyhedral cone σ. If $v, w \in \sigma$ and $v + w \in \tau$, then $v, w \in \tau$.

Proof. Classic. See [3] maybe. □

Proposition 3.1.16 (Dual cone of the intersection of halfspaces).

If $\sigma = H_{m_1}^+ \cap \cdots \cap H_{m_s}^+$, then

$$\sigma^\vee = \mathrm{Cone}(m_1, \dots, m_s).$$

Proof. Classic. See [3] maybe. □

Proposition 3.1.17 (Facets of a full dimensional cone).

If σ is a full dimensional cone, then facets of σ are of the form $H_m \cap \sigma$.

Proof. Classic. See [3] maybe. ☐

Proposition 3.1.18 (Intersection of facets containing a face)**.**
Every proper face $\tau \prec \sigma$ of a polyhedral cone σ is the intersection of the facets of σ containing τ.

Proof. Classic. See [3] maybe. ☐

Definition 3.1.19 (Dual face)**.**
Given a cone σ and a face $\tau \preceq \sigma$, the **dual face** to τ is

$$\tau^* := \sigma^\vee \cap \tau^\perp$$

Proposition 3.1.20 (The dual face is a face of the dual)**.**
If $\tau \preceq \sigma$, then $\tau^* \preceq \sigma^\vee$.

Proof. Classic. See [3] maybe. ☐

Proposition 3.1.21 (The double dual of a face)**.**
If $\tau \preceq \sigma$, then $\tau^{**} = \tau$.

Proof.
Classic. See [3] maybe. ☐

Proposition 3.1.22 (The dual of a face is antitone)**.**
If $\tau' \preceq \tau \preceq \sigma$, then $\tau' \preceq \tau$.

Proof. Classic. See [3] maybe. ☐

Proposition 3.1.23 (The dimension of the dual of a face)**.**
If $\tau \preceq \sigma$, then

$$\dim \tau + \dim \tau^* = \dim N.$$

Proof. Classic. See [3] maybe. ☐

3.1.3 Relative Interiors

Definition 3.1.24 (Relative interior)**.** The **relative interior**, aka **intrinsic interior**, of a cone σ is the interior of σ as a subset of its span.

Lemma 3.1.25 (The relative interior in terms of the inner product)**.**
For a cone σ,

$$u \in \mathrm{Relint}(\sigma) \iff \forall m \in \sigma^\vee \setminus \sigma^\perp, \langle m, u\rangle > 0$$

Proof. Classic. See [3] maybe. ☐

Lemma 3.1.26 (Relative interior of a dual face)**.**
If $\tau \preceq \sigma$ and $m \in \sigma^\vee$, then

$$m \in \mathrm{Relint}(\tau^*) \iff \tau = H_m \cap \sigma$$

Proof. Classic. See [3] maybe. ☐

Lemma 3.1.27 (Minimal face of a cone)**.**
If σ is a cone, then $W := \sigma \cap (-\sigma)$ is a subspace. Furthermore, $W = H_m \cap \sigma$ whenever $m \in \mathrm{Relint}(\sigma^\vee)$.

Proof. Classic. See [3] maybe. ☐

3.1.4 Strong Convexity

Definition 3.1.28 (Salient cones)**.** A cone σ is **salient**, aka **pointed** or **strongly convex**, if $\sigma \cap (-\sigma) = \{0\}$.

Proposition 3.1.29 (Alternative definitions of salient cones)**.**
The following are equivalent

1. *σ is salient*

2. $\{0\} \preceq \sigma$

3. *σ contains no positive dimensional subspace*

4. $\dim \sigma^\vee = \dim N$

Proof. Classic. See [3] maybe. □

3.1.5 Separation

Lemma 3.1.30 (Separation lemma)**.**
Let σ_1, σ_2 be polyhedral cones meeting along a common face τ. Then

$$\tau = H_m \cap \sigma_1 = H_m \cap \sigma_2$$

for any $m \in \mathrm{Relint}(\sigma_1^\vee \cap (-\sigma_2)^\vee)$.

Proof.
See [1]. □

3.1.6 Rational Polyhedral Cones

Let M and N be dual lattices with associated vector spaces $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R}, N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$.

Definition 3.1.31 (Rational cone)**.**
A cone $\sigma \subseteq N_\mathbb{R}$ is **rational** if $\sigma = \mathrm{Cone}(S)$ for some finite set $S \subseteq N$.

Lemma 3.1.32 (Faces of a rational cone)**.**
If $\tau \preceq \sigma$ is a face of a rational cone, then τ itself is rational.

Proof. Classic. See [3] maybe. □

Lemma 3.1.33 (The dual of a rational cone)**.**
$\sigma^\vee$ is a rational cone iff σ is.

Proof. Classic. See [3] maybe. □

Definition 3.1.34 (Ray generator)**.**
If ρ is an edge of a rational cone σ, then the monoid $\rho \cap N$ is generated by a unique element $u_\rho \in \rho \cap N$, which we call the **ray generator** of ρ.

Definition 3.1.35 (Minimal generators)**.**
The **minimal generators** of a rational cone σ are the ray generators of its edges.

Lemma 3.1.36 (A rational cone is generated by its minimal generators)**.**
A salient convex rational polyhedral cone is generated by its minimal generators.

Proof. Classic. See [3] maybe. □

Definition 3.1.37 (Regular cone)**.**
A salient rational polyhedral cone σ is **regular**, aka **smooth**, if its minimal generators form part of a $\mathbb{Z}$-basis of N.

Definition 3.1.38 (Simplicial cone)**.**
A salient rational polyhedral cone σ is **simplicial** if its minimal generators are $\mathbb{R}$-linearly independent.

3.1.7 Semigroup Algebras and Affine Toric Varieties

Definition 3.1.39 (Dual lattice of a cone)**.**
If $\sigma \subseteq N_{\mathbb{R}}$ is a polyhedral cone, then the lattice points

$$S_\sigma := \sigma^\vee \cap M$$

form a monoid.

Proposition 3.1.40 (Gordan's lemma)**.**
S_σ is finitely generated as a monoid.

Proof.
See [1]. □

Definition 3.1.41 (Affine toric variety of a rational polyhedral cone)**.**
$U_\sigma := \operatorname{Spec} \mathbb{C}[S_\sigma]$ is an affine toric variety.

Theorem 3.1.42 (Dimension of the affine toric variety of a rational polyhedral cone)**.**

$$\dim U_\sigma = \dim N \iff \textit{the torus of } U_\sigma \textit{ is } T_N = N \otimes_[\mathbb{Z}]\mathbb{C}^* \iff \sigma \textit{ is salient.}$$

Proof.
See [1]. □

Proposition 3.1.43 (The irreducible elements of the dual lattice of a cone)**.**
If $\sigma \subseteq N_{\mathbb{R}}$ is salient of maximal dimension, then the irreducible elements of S_σ are precisely the minimal generators of $\sigma^\vee$.

Proof.
See [1]. □

Chapter 4

Scheme theory

4.1 Correspondence between affine group schemes and Hopf algebras

We want to show that affine group schemes correspond to Hopf algebras. We must decide what this means mathematically.

We choose to interpret this as lifting Spec to a fully faithful functor from Hopf algebras to group schemes, with essential image affine group schemes.

An alternative would have been to lift the Gamma-Spec adjunction to an adjunction between Hopf algebras and affine group schemes. This is unfortunately much harder to do over an arbitrary scheme, so we leave this as future work.

4.1.1 Spec of an algebra

Definition 4.1.1 (Spec as a functor on algebras)**.** Spec is a contravariant functor from the category of R-algebras to the category of schemes over Spec_R.

Proposition 4.1.2 (Spec as a functor on algebras is fully faithful)**.**

Spec is a fully faithful contravariant functor from the category of R-algebras to the category of schemes over Spec_R*, preserving all limits.*

Proof.

$\mathrm{Spec} : \mathrm{Ring} \to \mathrm{Sch}$ is a fully faithful contravariant functor which preserves all limits, hence so is $\mathrm{Spec} : \mathrm{Ring}_R \to \mathrm{Sch}_{\mathrm{Spec}\, R}$ by Proposition 1.1.2 (alternatively, by Proposition 1.1.1). □

4.1.2 Spec of a bialgebra

Definition 4.1.3 (Spec as a functor on bialgebras)**.**

Spec is a contravariant functor from the category of R-bialgebras to the category of monoid schemes over Spec_R.

Proposition 4.1.4 (Spec as a functor on bialgebras is fully faithful)**.**

Spec is a fully faithful contravariant functor from the category of R-bialgebras to the category of monoid schemes over Spec_R.

Proof.

Spec : $\text{Ring}_R \to \text{Sch}_{\text{Spec}\, R}$ is a fully faithful contravariant functor preserving all limits according to Proposition 4.1.1, therefore Spec : $\text{Bialg}_R \to \text{GrpSch}_{\text{Spec}\, R}$ too is fully faithful according to 1.2.1. □

Proposition 4.1.5 (Spec sends cocommutative bialgebras to commutative monoid schemes)**.**
If A is a cocommutative bialgebra over R, then Spec A *is a commutative monoid scheme.*

Proof. Diagrams are the same up to identifying $\text{Spec}(A \otimes A)$ with $\text{Spec}\, A \otimes \text{Spec}\, A$. □

4.1.3 Spec of a Hopf algebra

Definition 4.1.6 (Spec as a functor on Hopf algebras)**.**

Spec is a contravariant functor from the category of R-Hopf algebras to the category of group schemes over Spec_R.

Proposition 4.1.7 (Spec as a functor on Hopf algebras is fully faithful)**.**
Spec is a fully faithful contravariant functor from the category of R-Hopf algebras to the category of group schemes over Spec_R.

Proof.

Spec : $\text{Ring}_R \to \text{Sch}_{\text{Spec}\, R}$ is a fully faithful contravariant functor preserving all limits according to Proposition 4.1.1, therefore Spec : $\text{Hopf}_R \to \text{GrpSch}_{\text{Spec}\, R}$ too is fully faithful according to 1.2.1. □

4.1.4 Essential image of Spec on Hopf algebras

Proposition 4.1.8 (Essential image of Spec on algebras)**.**
The essential image of Spec : $\text{Ring}_R \to \text{Sch}_{\text{Spec}\, R}$ *is precisely affine schemes over* $\text{Spec}\, R$*.*

Proof.

Direct consequence of Proposition 1.1.3. □

Proposition 4.1.9 (Essential image of Spec on bialgebras)**.**
The essential image of Spec : $\text{Bialg}_R \to \text{GrpSch}_{\text{Spec}\, R}$ *is precisely affine monoid schemes over* $\text{Spec}\, R$*.*

Proof.

Direct consequence of Propositions 1.2.3 and 4.1.8. □

Proposition 4.1.10 (Essential image of Spec on Hopf algebras)**.**
The essential image of Spec : $\text{Hopf}_R \to \text{GrpSch}_{\text{Spec}\, R}$ *is precisely affine group schemes over* $\text{Spec}\, R$*.*

Proof.

Direct consequence of Propositions 1.2.3 and 4.1.8. □

4.2 Diagonalisable groups

Definition 4.2.1 (The diagonalisable group scheme functor)**.**
Let G be a commutative group and S a base scheme. The diagonalisable group scheme $D_S(G)$ is defined as the base-change of $\operatorname{Spec} \mathbb{Z}[G]$ to S. For a commutative ring R, we write $D_R(G) := D_{\operatorname{Spec} R}(G)$.

Definition 4.2.2 (Diagonalisable group schemes)**.**
An algebraic group G over $\operatorname{Spec} R$ is **diagonalisable** if it is isomorphic to $D_R(G)$ for some commutative group G.

Lemma 4.2.3 (The diagonalisable group scheme torus over $\operatorname{Spec} R$)**.**
Let R be a commutative ring and M an abelian monoid. Then $D_R(M)$ is isomorphic to $\operatorname{Spec} R[M]$.

Proof. Ask any toddler on the street. □

Theorem 4.2.4.
An algebraic group G over a field k is diagonalizable if and only if $\Gamma(G)$ is spanned by its group-like elements.

Proof.
See Theorem 12.8 in [2]. □

Theorem 4.2.5.
Let R be a domain. The functor $D_R(G) := G \rightsquigarrow \operatorname{Spec} R[G]$ from the category of groups to the category of group schemes over $\operatorname{Spec} R$ is fully faithful.

Proof.
Compose Propositions 4.1.7 and 2.3.31.
Also see Theorem 12.9(a) in [2]. See SGA III Exposé VIII for a proof that works for R an arbitrary commutative ring. □

Proposition 4.2.6 (Morphisms between diagonalisable group schemes are affine)**.**
Let S be a scheme. Let M, N be commutative monoids and $f : M \to N$ a monoid hom. Then the map $D_S(f) : D_S(N) \to D_S(M)$ is affine.

Proof. $\operatorname{Spec} f : \operatorname{Spec} \mathbb{Z}[N] \to \operatorname{Spec} \mathbb{Z}[M]$ is affine, since it's a morphism of affine schemes. Therefore $D_S(f)$ is affine, as affine morphisms are preserved under base change. □

Proposition 4.2.7 (Closed embeddings between diagonalisable group schemes)**.**
Let S be a scheme. Let M, N be commutative monoids and $f : M \to N$ a surjective monoid hom. Then the map $D_S(f) : D_S(N) \to D_S(M)$ is a closed embedding.

Proof. Since f is surjective, the corresponding map $\hat{f} : \mathbb{Z}[M] \to \mathbb{Z}[N]$ is surjective too. Hence $\operatorname{Spec} \hat{f} : \operatorname{Spec} \mathbb{Z}[N] \to \operatorname{Spec} \mathbb{Z}[M]$ is a closed embedding. Therefore $D_S(f)$ is a closed embedding, as closed embeddings are preserved under base change. □

Proposition 4.2.8 (Faithfully flat morphisms between diagonalisable group schemes)**.**
Let S be a scheme. Let G, H be abelian groups and $f : G \to H$ an injective group hom. Then the map $D_S(f) : D_S(H) \to D_S(G)$ is faithfully flat.

Proof.

Since f is injective, $\mathbb{Z}[H]$ is a free module over $\mathbb{Z}[G]$ by Proposition 2.3.10, hence the map $\mathbb{Z}[G] \to \mathbb{Z}[H]$ is faithfully flat and so is $\operatorname{Spec} \mathbb{Z}[H] \to \operatorname{Spec} \mathbb{Z}[G]$. Therefore $D_S(f)$ is faithfully flat, as faithfully flat morphisms are preserved under base change. □

Proposition 4.2.9 (A subgroup of a diagonalisable group scheme is a diagonalisable group scheme)**.**

Let R be a domain. Let G be an abelian group. If H is a closed subgroup of $D_R(G)$, then H is a diagonalisable group scheme.

Proof.

H is a closed subscheme of an affine scheme, hence it is affine. By Proposition 4.1.10, write $H = \operatorname{Spec} A$ where A is a R-Hopf algebra. The closed subgroup embedding $H \hookrightarrow D_R(G)$ becomes a surjective bialgebra hom $R[G] \to A$ by Propositions 4.2.3 and 4.1.7. By Proposition 2.3.25, A is a diagonalisable bialgebra and therefore H is a diagonalisable group scheme. □

Proposition 4.2.10 (Diagonalisable group scheme of a torsion group is disconnected)**.**

Let G be an abelian group with an element of torsion n. Let R be a commutative ring with n invertible. Then $D_R(G)$ is disconnected.

Proof.

Say $x \in G$ is such that $x^n = 1$. Then

$$e : R[G] := \frac{1}{n} \sum_{i=0}^{n} x^i$$

is such that $e^2 = e$. We are done by Proposition 4.2.3. □

4.3 The torus

Definition 4.3.1 (The split torus)**.** The split torus $\mathbb{G}_m^n$ over a scheme S is the pullback of $\operatorname{Spec} \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ along the unique map $S \to \operatorname{Spec} \mathbb{Z}$.

Lemma 4.3.2 (Diag is a group isomorphism on hom sets)**.**

Let R be a domain. The functor $G \rightsquigarrow \operatorname{Spec} R[G]$ from the category of groups to the category of group schemes over $\operatorname{Spec} R$ is a group isomorphism on hom sets.

Proof.

Toddlers and streets by Lemmas 4.2.5 and 1.2.4. □

Definition 4.3.3 (Characters of a group scheme)**.**

For a group scheme G over S, the **character lattice** of G is

$$X(G) := \operatorname{Hom}_{\mathsf{GrpSch}_S}(G, \mathbb{G}_m).$$

An element of $X(G)$ is called a **character**.

Definition 4.3.4 (Cocharacters of a group scheme)**.**

For a group scheme G over S, the **cocharacter lattice** of G is

$$X^*(G) := \operatorname{Hom}_{\mathsf{GrpSch}_S}(\mathbb{G}_m, G).$$

An element of $X^*(G)$ is called a **cocharacter** or **one-parameter subgroup**.

Proposition 4.3.5 (Character lattice of a diagonalisable group scheme)**.**
Let R be a domain and G be a commutative group. Then $X(\operatorname{Spec} R[G]) = G$.

Proof.
By Propositions 4.2.3 and 4.2.5 in turn, we have

$$X(G) = \operatorname{Hom}_{\mathsf{GrpSch}}(G, \mathbb{G}_m) = \operatorname{Hom}(k[\mathbb{Z}], k[G]) = \operatorname{Hom}(\mathbb{Z}, G) = G.$$

□

Proposition 4.3.6 (Cocharacter lattice of a diagonalisable group scheme)**.**
Let R be a domain and G be a commutative group. Then $X^*(\operatorname{Spec} R[G]) = \operatorname{Hom}(G, \mathbb{Z})$.

Proof.
By Propositions 4.2.3 and 4.2.5 in turn, we have

$$X^*(G) = \operatorname{Hom}_{\mathsf{GrpSch}}(\mathbb{G}_m, G) = \operatorname{Hom}(k[G], k[\mathbb{Z}]) = \operatorname{Hom}(G, \mathbb{Z}).$$

□

Proposition 4.3.7 (Character lattice of the torus)**.**
Let G be a torus of dimension n over a domain R. Then $X(G) = \mathbb{Z}^n$.

Proof.
Immediate from Propositions 4.2.3 and 4.3.5. □

Proposition 4.3.8 (Cocharacter lattice of the torus)**.**
Let G be a torus of dimension n over a domain R. Then $X^*(G) = \operatorname{Hom}(\mathbb{Z}^n, \mathbb{Z})$.

Proof.
Immediate from Propositions 4.2.3 and 4.3.6. □

Definition 4.3.9 (The character-cocharacter pairing)**.**
Let R be a domain and G a group scheme over $\operatorname{Spec} R$. Then there is a $\mathbb{Z}$-valued perfect pairing between $X(G)$ and $X^*(G)$.

Proposition 4.3.10 (The character-cocharacter pairing is perfect)**.**
The character-cocharacter pairing on a torus is perfect.

Proof. Transfer everything across the isos $X(\mathbb{G}_m^n) = \mathbb{Z}^n, X * (\mathbb{G}_m^n) = \operatorname{Hom}(\mathbb{Z}^n, \mathbb{Z})$. □

Proposition 4.3.11 (The image of a torus is a torus)**.**
Let R be a domain. Let T be a split torus over R. Let G be a diagonalisable group scheme over R and let $\phi : T \to G$ be a homomorphism. Then the (scheme theoretic) image of ϕ is a split torus over R and the maps

$$T \xrightarrow{\hat{\phi}} \operatorname{im} \phi \xrightarrow{\iota} G$$

are group homomorphisms, and $\hat{\phi}$ is fpqc. Furthermore, if $T = D_R(H), G = D_R(I), \phi = D_R(f)$ for H a finitely generated free abelian group, I an abelian group, $f : I \to H$ a group hom, then $\operatorname{im} \phi \cong D_R(\operatorname{im}(f))$*.*

Proof.

By fullness of D_R (Proposition 4.2.5), it's enough to handle the case where $T = D_R(H), G = D_R(I), \phi = D_R(f)$ for H a finitely generated free abelian group, I an abelian group, $f : I \to H$ a group hom.

Then $\mathrm{im}(f)$ is a subgroup of the finitely-generated free abelian group H, hence itself a finitely-generated free abelian group (since free is equivalent to torsion-free for finitely-generated abelian groups, and a subgroup of a torsion-free group is torsion-free). □

Proposition 4.3.12 (A subgroup of a torus is a torus)**.**

Let R be a commutative ring of characteristic zero. Let T be a split torus. If $H \subseteq T$ is a connected closed subgroup, then H is a split torus.

Proof.

By assumption, write $T \cong D_k[G]$ for G a free abelian group. By Proposition 4.2.9, H is a diagonalisable group scheme, say $H \cong D_k(I)$ for I an abelian group. Since H is a closed subscheme, the map $G \to I$ is surjective, so I is a finitely-generated abelian group. Since H is connected, Proposition 4.2.10 says I is torsion-free, hence free. Thus H is a split torus. □

Chapter 5

Toric varieties

5.1 Toric varieties

In this section, we define toric varieties and toric morphisms.

Definition 5.1.1 (Toric varieties)**.**
Let k be a field. Let T be a torus over k. A *toric variety* structure on a scheme X over k consists the following data:

- a torus T over k,
- a group action $T \times X \to X$ over k.
- a dominant open immersion $i : T \hookrightarrow X$ over k that is T-equivariant.

Definition 5.1.2 (Torus morphisms, torus isomorphisms)**.**
Let k be a field. Let T_1, T_2 be tori over k. Let X_1, X_2 be toric varieties with torus T_1, T_2 respectively. A *toric morphism* from X_1 to X_2 is the data of a k-morphism $X_1 \to X_2$ and a k-group homomorphism $T_1 \to T_2$ that commute with the embeddings $T_1 \to X_1, T_2 \to X_2$ and the actions. A *toric isomorphism* from X_1 to X_2 is the data of two isomorphisms $X_1 \cong X_2$ and $T_1 \cong T_2$ that commute with the embeddings $T_1 \to X_1, T_2 \to X_2$.

5.2 Affine toric varieties and affine monoids

In this section, we construct affine toric varieties from affine monoids, and show all affine toric varieties arise from affine monoids in this way.

5.2.1 Toric varieties from affine monoids

Proposition 5.2.1 (The diagonalisable group scheme of an affine monoid algebra is an affine toric variety)**.**
Let k be a field. Let G be a finitely generated free abelian group. Let M be an affine monoid with Grothendieck group G. Then $D_k(M)$ is an affine toric variety over k with torus $D_k(G)$.

Proof.
The map $D_k(G) \hookrightarrow D_k(M)$ is given by the embedding $M \hookrightarrow G$.
We identify $D_k(G) \cong \operatorname{Spec} k[G], D_k(M) \cong \operatorname{Spec} k[M]$ using Proposition 4.2.3.

By Proposition 2.3.12, $k[G]$ is a localization of $k[M]$. Therefore the map $\operatorname{Spec} k[G] \hookrightarrow \operatorname{Spec} k[M]$ is an open immersion.

This open immersion is dominant since $\operatorname{Spec} k[M]$ is irreducible as $k[M]$ is a domain (Proposition 2.2.4).

The group action $D_k(G) \times D_k(M) \to D_k(M)$ comes from pulling back along $D_k(G) \hookrightarrow D_k(M)$ the left action $D_k(M) \times D_k(M) \to D_k(M)$ using Proposition 1.2.5. □

5.2.2 Essential surjectivity from affine monoids to affine toric varieties

Definition 5.2.2 (The character eigenspace)**.**

For a finite dimensional representation of a torus T on W, the **character eigenspace** of a character $\chi \in X(T)$ is

$$W_m = \{w \in W : t \cdot w = \chi(t) \text{ for all } t \in T\}.$$

Proposition 5.2.3 (Decomposition into character eigenspaces)**.**

The space decomposes into the direct sum of the character eigenspaces.

Proof. TODO □

Definition 5.2.4.

There is a torus action on the semigroup algebra $\mathbb{C}[M]$: given $t \in T_N$ and $f \in \mathbb{C}[M]$ define

$$t \cdot f = (p \mapsto f(t^{-1}p)).$$

Lemma 5.2.5. *Let $A \subseteq \mathbb{C}[M]$ be a stable subspace, then*

$$A = \bigoplus_{\chi^m \in A} \mathbb{C} \cdot \chi^m.$$

Proof.

TODO □

Definition 5.2.6 (Characters of a toric variety)**.** Let k be a field. Let T be a torus over k. Let V be a toric variety with torus k. The *characters* $X(V)$ of V are defined as the intersection of $X(T)$ with the image of the map $k[V] \to k[T]$ of coordinate rings induced by the embedding $T \hookrightarrow V$.

Proposition 5.2.7 (Characters of a toric variety are an affine monoid)**.**

Let k be a field. Let T be a torus over k. Let V be a toric variety with torus k. Then $X(V)$ is an affine monoid.

Proof. TODO □

5.3 Affine toric varieties and toric ideals

In this section, we define toric ideals, show that one can construct toric varieties from them and that all toric varieties arise in this way.

5.3.1 Toric ideals and affine monoids

Definition 5.3.1 (Lattice ideal)**.** Let R be a ring. Let G be a free abelian group and M an affine monoid whose Grothendieck group is G. Let $L \leq G$ be a sublattice. The *lattice ideal* of L is the R-ideal of $R[M]$ defined by

$$I_L := \langle X^\alpha - X^\beta | \alpha, \beta \in M, \alpha - \beta \in L \rangle.$$

Definition 5.3.2.

Let R be a ring. Let M be an affine monoid. A *toric ideal* is a prime lattice R-ideal of $R[M]$.

Proposition 5.3.3 (An ideal is toric iff it is prime and generated by binomials)**.**

An ideal is toric if and only if it's prime and generated by binomials $X^\alpha - X^\beta$.

Proof.

A toric ideal is prime and generated by binomials by definition.

Assume I is prime and generated by $X^\alpha - X^\beta$ ranging over $(\alpha, \beta) \in S$ for some set $S \subseteq M \times M$.

Note first that I doesn't contain any monomial. Indeed, I is contained in the kernel of the map $R[M] \to R$ given by $X^m \mapsto 1$.

Since I is prime, this means that

$$X^{\alpha+\gamma} - X^{\beta+\gamma} \in I \iff X^\alpha - X^\beta \in I.$$

In particular, if $\alpha_1 - \beta_1 = \alpha_2 - \beta_2$, then Since I is prime, this means that

$$X^{\alpha_1} - X^{\beta_1} \in I \iff X^{\alpha_1+\beta_2} - X^{\beta_1+\beta_2} \in I \iff X^{\alpha_2+\beta_1} - X^{\beta_1+\beta_2} \in I \iff X^{\alpha_2} - X^{\beta_2} \in I.$$

Now, we claim that $I = I_L$ where $L \leq G$ is given by

$$\operatorname{span}\{\delta - \varepsilon | (\delta, \varepsilon) \in S\}.$$

Clearly, $I \subseteq I_L$.

For the other direction, assume $\delta, \varepsilon \in M, \delta - \varepsilon \in L$. Let's prove $X^\delta - X^\varepsilon \in I$ by induction on $\delta - \varepsilon \in L$:

- If $\delta - \varepsilon = 0$, then $X^\delta - X^\varepsilon = 0 \in I$.
- If $\delta_1 - \varepsilon_1 = \varepsilon_2 - \delta_2$ and $X^{\delta_2} - X^{\varepsilon_2} \in I$, then

  $$X^{\delta_1} - X^{\varepsilon_1} \in I \iff X^{\varepsilon_2} - X^{\delta_2} \iff X^{\delta_1} - X^{\varepsilon_1}$$

  by the remark, and this holds by assumption.
- If $\delta - \varepsilon = \alpha - \beta$ where $(\alpha, \beta) \in S$, then

  $$X^\delta - X^\varepsilon \in I \iff X^\alpha - X^\beta \in I$$

  by the remark, and this holds by assumption.
- Assume $\delta_1, \delta_2, \varepsilon_1, \varepsilon_2$ are such that $X^{\delta_1} - X^{\varepsilon_1}, X^{\delta_2} - X^{\varepsilon_2} \in I$. Then

  $$X^{\delta_1+\delta_2} - X^{\varepsilon_1+\varepsilon_2} = (X^{\delta_1} - X^{\varepsilon_1})X^{\delta_2} + X^{\varepsilon_1}(X^{\delta_2} - X^{\varepsilon_2}) \in I.$$

□

Proposition 5.3.4 (The vanishing ideal of a closed toric embedding)**.**

Let k be a field. Let V be a toric variety over k. Let $i : V \hookrightarrow \mathbb{A}^n$ be a closed toric embedding. Then the vanishing ideal of i is toric.

Proof.

TODO □

5.4 The $Y_{\mathcal{A}}$ construction

The $Y_{\mathcal{A}}$ construction is an alternative construction to the one we use in Section 5.2. Morally, the difference is that $Y_{\mathcal{A}}$ is "extrinsic" while our construction is "intrinsic". As a result, our construction is canonical, while $Y_{\mathcal{A}}$ isn't. $Y_{\mathcal{A}}$ is still useful to study toric ideals, but we do not need it in Toric.

Definition 5.4.1.
Let S be a scheme. Let G be an abelian group. Let s be an arbitrary indexing type, and $\mathcal{A} : s \to G$ an indexed family. Let $f'_{\mathcal{A}}$ be the map

$$\begin{aligned} \mathbb{Z}^{\oplus s} &\to G \\ e_i &\mapsto \mathcal{A}_i. \end{aligned}$$

and define $\Phi'_{\mathcal{A}} : D_S(G) \to \mathbb{G}_m^s$ as the image under D_S of $f'_{\mathcal{A}}$.

Definition 5.4.2.
Let S be a scheme. Let G be an abelian group. Let s be an arbitrary indexing type, and $\mathcal{A} : s \to G$ an indexed family. Let $f_{\mathcal{A}}$ be the map

$$\begin{aligned} \mathbb{N}^{\oplus s} &\to G \\ e_i &\mapsto \mathcal{A}_i. \end{aligned}$$

and define $\Phi_{\mathcal{A}} : D_S(G) \to \mathbb{A}^s$ as the image under D_S of $f_{\mathcal{A}}$.

Definition 5.4.3.
$Y_{\mathcal{A}}$ is the scheme theoretic closure of $\operatorname{im} \Phi_{\mathcal{A}}$ in $\mathbb{A}^s$.

Proposition 5.4.4.
Let the base be $S = \operatorname{Spec} k$ for a field k, then $Y_{\mathcal{A}}$ is a toric variety.

Proof.
Torus: Define the torus T' to be the one we get from 4.3.11 with quotient map $\pi : T \to T'$. **O**pen embedding: Since both $Y_{\mathcal{A}}, T'$ are closures of Φ, Φ' and $\mathbb{G}_m^n \to \mathbb{A}^n$ is an open embedding we get an open embedding $\iota : T' \to Y_{\mathcal{A}}$ such that the map $\phi : T \to Y_{\mathcal{A}}$ factors as $\phi = \iota \circ \pi$. **D**ominant: Since ϕ is dominant, so is ι. **A**ction: Since $\mathbb{G}_m^n$ acts on $\mathbb{A}^n$, we get a morphism $a' : T' \times_S Y_{\mathcal{A}} \to \mathbb{A}^n$. As $T' \times_S Y_{\mathcal{A}}$ is reduced (TODO add lemma), to show that this factors through $Y_{\mathcal{A}}$ it suffices to check that the image lies in $Y_{\mathcal{A}}$.

First, $T' \times_S T' \to T' \times Y_{\mathcal{A}}$ is dominant, since T' is flat and flat base change preserves dominance (TODO add lemma). Since the image of $T' \times_S T'$ is T' we're done for topological reasons.

Equivariant: The inclusion of the torus is equivariant, since $\mathbb{G}_m^n \to \mathbb{A}^n$ is. □

Proposition 5.4.5.
The character lattice of the torus of $Y_{\mathcal{A}}$ is $\mathbb{Z}\mathcal{A}$.

Proof.
$\Phi_{\mathcal{A}} : T_N \to \mathbb{G}_m^s$ factors through the torus of $Y_{\mathcal{A}}$. The conclusion follows from looking at the corresponding maps of character lattices. □

Proposition 5.4.6.
Let R be a commutative ring. Let G be an abelian group. Let s be an arbitrary indexing type, and $\mathcal{A} : s \to G$ an indexed family. Let L be the kernel of $f'_{\mathcal{A}} : \mathbb{Z}^{\oplus s} \to G$. Then the ideal of the affine toric variety $Y_{\mathcal{A}}$ is

$$I(Y_{\mathcal{A}}) = I_L.$$

Proof.

By the definition of $Y_{\mathcal{A}}$ as the scheme-theoretic closure of $\Phi_{\mathcal{A}}$, we have $I(Y_{\mathcal{A}}) = \ker R[f_{\mathcal{A}}] = \ker R[f'_{\mathcal{A}}]$ where, recall, $f_{\mathcal{A}} : \mathbb{N}^s \to G, f'_{\mathcal{A}} : \mathbb{Z}^s \to G$ are both given by $e_i \mapsto \mathcal{A}_i$, and $R[f_{\mathcal{A}}] : R[\mathbb{N}^s] \to R[G]$ is the pushforward.

By Proposition 2.3.11 with $G := \mathbb{Z}, S := 1$,

$$\ker R[f_{\mathcal{A}}] = \operatorname{span}\{X^\alpha - X^\beta | \alpha, \beta \in \mathbb{N}^s, f(\alpha) = f(\beta)\} = \operatorname{span}\{X^\alpha - X^\beta | \alpha, \beta \in \mathbb{N}^s, \alpha - \beta \in L\}.$$

□

Proposition 5.4.7.

If S is an affine monoid and $\mathcal{A}$ is a finite set generating S as a monoid, then $\operatorname{Spec} \Bbbk[S] = Y_{\mathcal{A}}$.

Proof.

We get a $\Bbbk$-algebra homomorphism $\pi : \Bbbk[x_1, \dots, x_s] \to \Bbbk[\mathbb{Z}S]$ given by $\mathcal{A}$; this induces a morphism $\Phi_{\mathcal{A}} : T \to \Bbbk^s$. The kernel of π is the toric ideal of $Y_{\mathcal{A}}$ and π is clearly surjective, so $Y_{\mathcal{A}} = \mathbb{V}(\ker(\pi)) = \operatorname{Spec} \Bbbk[x_1, \dots, x_s]/\ker(\pi) = \operatorname{Spec} \mathbb{C}[S]$. □

Proposition 5.4.8.

The ideal of $Y_{\mathcal{A}}$ is a toric ideal.

Proof.

Immediate consequence of Proposition 5.4.6. □

Theorem 5.4.9 (Affine toric varieties come from affine monoids)**.**

Let k be a field. Let T be a torus over k. Let V be a toric variety with torus k. Then there exists a torus isomorphism $V \cong D_k(X(V))$.

Proof.

TODO □

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